1. The random variables $\mathbf{X}$ and $\mathbf{Y}$ have joint pdf given by

$$f_{X,Y}(u,v) = \begin{cases} 2\exp -(u+v) & 0 < u < v < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Are the random variables $\mathbf{X}$ and $\mathbf{Y}$ independent ?

(b) Find $P\{\mathbf{Y} > 3\mathbf{X}\}$.

(c) For $\alpha > 0$, find $P\{\mathbf{X} + \mathbf{Y} \le \alpha\}$.

(d) Use the result in (c) to determine the pdf of the random variable $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$.

(e) An alternative method for computing $f_Z(w)$ is to use the magical mystical integral formula $f_Z(w) = \int_{-\infty}^{\infty} f_{X,Y}(u, w-u)du = \int_{-\infty}^{\infty} f_{X,Y}(w-v, v)dv$. Use either integral to compute $f_Z(w)$ and compare with the result obtained in part (d).

(f) Find the pdf of $\max(\mathbf{X}, \mathbf{Y})$. Think before you write (which is a suggestion that you should consider taking to heart in general as well as in this particular case!).

2. Ross, #8, p.293 (p. 297 in the 4th edition)

3. Let $(\mathbf{X}, \mathbf{Y})$ have joint pdf $f_{\mathbf{X},\mathbf{Y}}(u,v) = \begin{cases} c(1 - \sqrt{u^2+v^2}), & 0 < u^2+v^2 < 1, \\ 0, & \text{otherwise.} \end{cases}$

(a) What is the value of the constant c?

(b) Let α denote some number between 0 and 1. What are the values of $P\{\sqrt{\mathbf{X}^2+\mathbf{Y}^2} < \alpha\}$ and $P\{\mathbf{X}^2+\mathbf{Y}^2 < \alpha\}$?

(c) Find the pdf of the random variable $\mathbf{R} = \sqrt{\mathbf{X}^2+\mathbf{Y}^2}$ and the pdf of the random variable $\mathbf{Z} = \mathbf{R}^2 = \mathbf{X}^2+\mathbf{Y}^2$. Be sure to specify the pdf for all values of the argument from $-\infty$ to $+\infty$.

(d) Prove that your answers of part (c) are indeed valid pdfs.

4. Let $(\mathbf{X},\mathbf{Y})$ be uniformly distributed on the interior of the square with vertices at (1,0), (0,1), (–1,0), and (0,–1).

(a) Find the joint pdf of $\mathbf{X}+\mathbf{Y}$ and $\mathbf{X}-\mathbf{Y}$.

(b) Determine whether $\mathbf{X}$ and $\mathbf{Y}$ are uncorrelated or independent.

5. Let $(\mathbf{X},\mathbf{Y})$ be uniformly distributed on the unit disc (radius = 1) centered at the origin of a two dimensional plane. Find the expected value of the distance from the origin to the point $(\mathbf{X},\mathbf{Y})$.

6. $\mathbf{X}$ and $\mathbf{Y}$ are independent random variables uniformly distributed on [0,1].

(a) Find the pdf of $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$.

(b) Let $\mathbf{A} = \min(\mathbf{X},\mathbf{Y})$ and $\mathbf{B} = \max(\mathbf{X},\mathbf{Y})$. Use the results on p. 146 of the Lecture Notes to write down the joint pdf $f_{\mathbf{A},\mathbf{B}}(\alpha,\beta)$. You *should* get a pdf that we have studied in class except, of course, that α and β are being used in place of u and v.

(c) In class, we also found the pdf of $\mathbf{A} + \mathbf{B}$ when the joint pdf is as in part (b). Why is the pdf of $\mathbf{A} + \mathbf{B}$ so remarkably similar to the pdf of $\mathbf{Z}$?

7. In the Lecture Notes, it is alleged on page 146 that if $\mathbf{X}$ and $\mathbf{Y}$ are independent exponential random variables with parameter 1, then $\mathbf{W} = \mathbf{X} + \mathbf{Y}$ and $\mathbf{Z} = \mathbf{X}^2 + \mathbf{Y}^2$ have joint pdf

$$f_{\mathbf{W},\mathbf{Z}}(\alpha,\beta) = \begin{cases} \dfrac{\exp(-\alpha)}{\sqrt{2\beta - \alpha^2}}, & 0 < \sqrt{\beta} < \alpha < \sqrt{2\beta} \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Draw a sketch of the plane with axes α and β, and indicate the region over which the joint pdf is nonzero.

(b) Explain why the joint pdf is zero for $\alpha < \sqrt{\beta}$.

(c) Find the marginal pdf of **W** by integrating $f_{\mathbf{W},\mathbf{Z}}(\alpha,\beta)$ with respect to β.

(d) Use Proposition 3.1 (p. 266, or p. 271 in the 4th edition) of Ross to *deduce* what the pdf of **W** = **X** + **Y** *ought* to be, and compare the result to your answer of part (c). Are they the same or different? Explain.

8. Except for the trivial case when all the probability mass is at $\mu = E[\mathbf{X}]$, there is probability mass both to the left and right of μ; in particular, there is an ω Ω such that $\mathbf{X}(\omega) < \mu$. Is it also true that if $(E[\mathbf{X}], E[\mathbf{Y}]) = (\mu_1, \mu_2)$, then there is an ω Ω such that $\mathbf{X}(\omega) < \mu_1$ and $\mathbf{Y}(\omega) < \mu_2$? If you believe the result is true, prove it. Otherwise, give a counterexample to show that it is false.

9. Let the random variables **X** and **Y** be independent and uniformly distributed on (0,1). Find $E(|\mathbf{X}-\mathbf{Y}|)$ and $\mathrm{Var}(\mathbf{X}-\mathbf{Y})$.

10. Let $E[\mathbf{X}] = 1$, $E[\mathbf{Y}] = 4$, $\mathrm{var}(\mathbf{X}) = 4$, $\mathrm{var}(\mathbf{Y}) = 9$, and $\rho_{\mathbf{X},\mathbf{Y}} = 0.1$

(a) If $\mathbf{Z} = 2(\mathbf{X}+\mathbf{Y})(\mathbf{X}-\mathbf{Y})$, what is $E[\mathbf{Z}]$?

(b) If $\mathbf{T} = 2\mathbf{X}+\mathbf{Y}$ and $\mathbf{U} = 2\mathbf{X}-\mathbf{Y}$, what is $\mathrm{cov}(\mathbf{T}, \mathbf{U})$?

(c) If $\mathbf{W} = 3\mathbf{X} + \mathbf{Y} + 2$, find $E[\mathbf{W}]$ and $\mathrm{var}(\mathbf{W})$.

(d) If **X** and **Y** are jointly Gaussian random variables, and **W** is as defined in (c), what is $P\{\mathbf{W} > 0\}$?

11.(a) If $\mathrm{var}(\mathbf{X} + \mathbf{Y}) = 36$ and $\mathrm{var}(\mathbf{X} - \mathbf{Y}) = 64$, what is $\mathrm{cov}(\mathbf{X}, \mathbf{Y})$? If you are also told that $\mathrm{var}(\mathbf{X}) = 3 \bullet \mathrm{var}(\mathbf{Y})$, what is $\rho_{\mathbf{X},\mathbf{Y}}$?

(b) If instead of having values 36 and 64, $\mathrm{var}(\mathbf{X}) = \mathrm{var}(\mathbf{Y})$, are **X** and **Y** uncorrelated ?

12. Two random break–points $\mathbf{X}_1$ and $\mathbf{X}_2$ are chosen on a stick of unit length, thereby breaking the stick into three pieces. Thus, if $\mathbf{X}_2 > \mathbf{X}_1$, the three pieces have lengths $\mathbf{X}_1$, $\mathbf{X}_2 - \mathbf{X}_1$, and $1 - \mathbf{X}_2$. (If $\mathbf{X}_1 > \mathbf{X}_2$, interchange $\mathbf{X}_1$ and $\mathbf{X}_2$ in the above. Luckily, $P\{\mathbf{X}_1 = \mathbf{X}_2\} = 0$).

(a) Sketch the u–v plane and indicate on it the region(s) such that if the random point $(\mathbf{X}_1, \mathbf{X}_2)$ lies in the region(s), then the pieces can form a triangle.

(b) Assume that the random variables $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent and uniformly distributed on (0,1). Show that the desired probability is 1/4 in this case by integrating the joint pdf over the region(s) found in part (a).

(c) Suppose that $\mathbf{X}_1$ is uniformly distributed on (0,1) and that the *conditional* density of $\mathbf{X}_2$ given $\mathbf{X}_1 = u$ is uniform on (u,1), that is, we break the stick at $\mathbf{X}_1$ (choosing the break-point with uniform density), and then break the right-hand piece at $\mathbf{X}_2$ (choosing the break-point with uniform density on the *right-hand piece).* Let T denote the event that the pieces can form a triangle. Find the *conditional probability* $P(T|\mathbf{X}| = u)$, for $0 < u < 1$. (Hint: you will find that different expressions apply depending on the value of u). Now find the *unconditional probability* using the result that

$$P(T) = \int P(T|\mathbf{X}_1 = u) f_{\mathbf{X}_1}(u) du$$

where the integral is over the range $(-\infty, \infty)$ in general (but only over (0,1) in this case).

(d) *With the same assumptions as in (c),* now find the joint pdf of $\mathbf{X}_1$ and $\mathbf{X}_2$. Show that this joint pdf is nonzero only over the triangular region $0 < u < v < 1$. Now, use the method used in part (b) to find the probability that the pieces can form a triangle. Is the answer the same as in part (c) ?

(e) From the joint pdf in (d), compute the marginal pdfs of $\mathbf{X}_1$ and $\mathbf{X}_2$. The marginal pdf of $\mathbf{X}_1$ should be uniform on (0,1). (It isn't? Better check your work!). **Prove** that the pdf that you obtain for $\mathbf{X}_2$ is a valid pdf.

Exercise: ("Sticks and stones may break my bones, but more exercises in probability can never hurt me!") What if after breaking the stick at $\mathbf{X}_1$, we pick one of the pieces at random and break *it* at random? Thus, given $\mathbf{X}_1 = u$, the conditional pdf of $\mathbf{X}_2$ is uniform on (0, u) or (u, 1) depending on which piece is picked. What is P(T) in this case?

13. The Sirrah Poll wishes to assess the popularity of Knute Gingpoor. To this end, a random sample of n persons is asked for opinions, with the opinion of the i–th person being denoted by $\mathbf{X}_i$ where $\mathbf{X}_i = 1$ if the person supports Knute, and $\mathbf{X}_i = 0$ if the person does not support Knute. The Sirrah Poll treats the $\mathbf{X}_i$'s as *independent* random variables with $P\{\mathbf{X}_i = 1\} = p$ for all i, and estimates p as $(\sum \mathbf{X}_i)/n$. The Poll wishes to be *fairly sure* that its estimate of p has a margin of error of 2% or less. The Poll thus wants to have the following inequality hold:

$$P\{|(\sum \mathbf{X}_i)/n - p| > 0.02\} \leq 0.05.$$

In other words, with high probability (0.95), the *estimate of p* differs from the *actual value of p* by at most 0.02 (2%). That evening, the networks announce that the Sirrah Poll had found that Knute Gingpoor has a popularity rating of $100[(\sum \mathbf{X}_i)/n]\%$ and that the margin of error of the poll is ±2%.

(a) Suppose that p = 0.1. Use the weak law of large numbers to find N such that for all n ≥ N, the above inequality is guaranteed to hold. Thus, the Poll should have used a random sample of size N or more. What if p = 0.2 ? 0.3 ? . . .

(b) Express N as a function of p. What is the maximum value of this function?

(c) The Sirrah Poll naturally wishes to minimize the number of persons surveyed in order to minimize the cost. However, the value of p is unknown. How many voters should the Poll survey so that, *regardless of the value of p*, the above inequality will be satisfied?

14.(a) A fair die is rolled. Let $\mathbf{X}$ denote the outcome (i.e. the number showing). What is the mean and variance of $\mathbf{X}$?

(b) The die is rolled 1000 times and the 1000 outcomes are added together. The result is denoted by $\mathbf{Y}$. What is the minimum value of $\mathbf{Y}$? What is the maximum value?

(c) Estimate $P\{\mathbf{Y} < 3500\}$.