Assigned: Wednesday, November 26, 1997
Due: Wednesday, December 3, 1997
Reading: Ross, Chapters 5 and 6
Suggested Noncredit Exercises: pp. 232–237: 12, 13, 15–19, 21, 27, 29,30–39; pp. 237–241: 13–22, 24, 26, 28, 29; pp. 241-243: 3, 4, 8, 10, 13, 14-16
Noncredit Exercises: Those using the 4th edition should try Ross pp. 239-243: 13-22, 24, 26, 28, 29; pp. 243-248: 12, 13, 15-19, 21, 27, 29, 30-39.
Problems:

1. The radius of a sphere is a random variable **R** with pdf $f_{\mathbf{R}}(\rho) = \begin{cases} 3\rho^2, & 0 < \rho < 1, \\ 0 & \text{elsewhere.} \end{cases}$

(a) Use LOTUS to find the average radius, average volume and average surface area of the sphere. Does a sphere of average radius have average volume? Does a sphere of average radius have average surface area?

(b) On Problem 5 of Problem Set #10, we found the pdf of the volume **V**. Find E[**V**] directly from this pdf. Do you get the same answer as in part (a)? Why or why not?

2. Raw scores on the SAT (and GRE) are transformed by a nonlinear function so that the minimum score is 200 and the maximum is 800. The histogram of scores *resembles* a Gaussian pdf with mean 500 and variance $\sigma^2 = 100^2$, that is, the score **X** of a student chosen at random can be modeled as a Gaussian random variable with mean 500 and variance $\sigma^2 = 100^2$. According to this model,

(a) what should your percentile rank be if your score is 700?

(b) what score corresponds to a percentile rank of 95% ?

(c) What fraction of students score between 300 and 550?

3. Let $Q(x) = \int_x^\infty (\sqrt{2\pi})^{-1} \exp\left(-\frac{u^2}{2}\right) du = 1 - \Phi(x)$ where $\Phi(x)$ denotes the CDF of a unit Gaussian random variable.

(a) Some tables list the values of $Q(x)$ (instead of $\Phi(x)$) for large values of x. Why might the tabulator have chosen to specify $Q(x)$ instead of $\Phi(x)$? Explain briefly.
On page 211 (p. 218 in 4th edition), Ross gives an upper and a lower bound on $Q(x)$ (Eq. (4.4)). The rest of this problem leads you through a derivation of Eq. (4.4) that does not use the “obvious inequality” invoked by Ross in his proof, and it also looks at another, simpler bound.

(b) What is the derivative of $\exp(-u^2/2)$ with respect to u?

(c) Write the integrand for $Q(x)$ as $(\sqrt{2\pi})^{-1}u^{-1}(u\exp(-u^2/2))$ and integrate by parts to deduce the upper bound on $Q(x)$. Repeat the trick of re–writing and integrating by parts to deduce the lower bound on $Q(x)$. Are these bounds useful as $x \to 0$? Why or why not? What is the asymptotic value of the ratio of the bounds as $x \to \infty$?

(d) A useful bound when x is small is $Q(x) \le (1/2)\exp(-x^2/2)$ for $x \ge 0$ in which equality holds only at x = 0. Derive this bound by first showing that $t^2 - x^2 > (t - x)^2$ for $t > x > 0$ and then applying this result to $\exp(x^2/2)Q(x) = \int_x^\infty (\sqrt{2\pi})^{-1} \exp\left[-\left(\frac{t^2 - x^2}{2}\right)\right] dt$

(e) For what values of x is this smaller than the upper bound of Eq.(4.4)?

4. Do **either** part (a) **or** part (b). Then do parts (c)–(e).

(a) Include with your homework a photocopy of your calculator’s manual page(s) that explains which **formula** your calculator computes Q(x). Reading the page might help too!
Note: I **do not want** to know **which buttons** you have to press in order to find Q(x); I **want** to know **what formula** your calculator uses internally to find Q(x).

The xerographically–challenged are permitted to just copy the relevant formulas to their homework. Press the appropriate buttons to find Q(5).
If your calculator cannot compute Q(x), or if the manual does not state what formula is used to calculate Q(x) but just tells you which buttons to press, or if you have lost the manual, do part (b) instead.

(b) Read Chapter 26.2 of Abramowitz and Stegun (*reference* book *(not a* ***reserve*** *book)* in Grainger Engineering Library), and use Equation 26.2.17 to calculate Q(5).

(c) The number found in part (a) or (b) is just an *approximation* to the value of Q(5). Use the maximum error specification to find the *range* in which the actual value of Q(5) must necessarily lie. What is the *maximum relative error* in the approximation to Q(5) that you found in part (a) or (b)? Note: the relative error is defined as $\frac{|\text{true value} - \text{computed value}|}{\text{true value}}$ expressed as a percentage.

(d) On p. 972, Abramowitz and Stegun give the value of $-\log_{10}Q(5)$. Blindly trust your calculator to do the exponentiation correctly and find the *actual relative error* in the approximation to Q(5) that you found in part (a) or (b). What would the actual relative error have been if you had simply used the upper bound of Eq. (4.4) as an approximation to Q(5) as suggested by Ross? What if you had ignored Ross's suggestion and used the lower bound as an approximation to Q(5) instead?

(e) Explain why the "much easier" Equation 26.2.18 of Abramowitz and Stegun is not particularly useful for computing Q(5).

5. A Rayleigh random variable **X** has pdf $f(u) = (u/ß^2)\exp(-u^2/2ß^2)$ for $u > 0$ and complementary CDF given by $P\{\mathbf{X} > t\} = \exp(-t^2/2ß^2)$ for $t > 0$.

(a) Use the result $E[\mathbf{X}] = \int_0^\infty P\{\mathbf{X} > t\}dt$ (Ross, Lemma 2.1, p. 197; p. 203 in the 4th edition) to find the mean lifetime of the chip.

(b) Use the result that $E[\mathbf{X}] = \int_0^\infty uf(u)du$ to find the mean lifetime of the chip. Do you get the same answer as in part (a)? Why or why not?

(c) What is the median lifetime of the chip, and is it larger or smaller than the mean lifetime?

6. An IC module has constant hazard rate $\lambda = -\ln 0.999$/week.

(a) What is the average lifetime (in weeks)? What is the median lifetime?

(b) What is the probability that the module lasts for at least one week?
Now suppose that three identical modules are organized into a triple–modular–redundancy (TMR) system in which we assume that the majority–logic gate cannot fail. Furthermore, we assume that the three modules fail independently of one another, that is, if their lifetimes are $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$, then the events $\{\mathbf{X}_1 > t_1\}$, $\{\mathbf{X}_2 > t_2\}$, and $\{\mathbf{X}_3 > t_3\}$ are independent for all t_1, t_2, t_3. Let **Y** denote the length of time for which the TMR system functions correctly.

(c) If you are told that the event $\{\mathbf{Y} > t\}$ occurred, what can you say about the occurence (or nonoccurence) of the events $\{\mathbf{X}_1 > t\}$, $\{\mathbf{X}_2 > t\}$, and $\{\mathbf{X}_3 > t\}$?

(d) Show that $P\{\mathbf{Y} > t\} = 3\exp(-2\lambda t) - 2\exp(-3\lambda t)$ and use this result to find the average lifetime and the median lifetime of the TMR system. Compare your answers to those in part (a). Do the results surprise you? Is the TMR system improving performance the way it is alleged to?

(e) What is the probability that the TMR system functions correctly for at least one week? Compare this answer to that of part (b). Do you think that the TMR system is more reliable or less reliable?

(f) Find t such that $P\{\mathbf{Y} > t\} = 0.999$ and compare the answer to that of part (b). Has the TMR system improved performance?