Assigned: Wednesday, October 29, 1997
Due: Wednesday, November 5, 1997
Reading: Ross, Chapters 4 and 5
Suggested Noncredit Exercises: Ross, pp. 173-184: 2, 7, 13, 17, 20, 25, 37, 39, 41-45; pp. 184-188: 10-13; pp. 232-237: 1-8; pp. 237-241: 1, 8
Those using the 4th edition should try Ross pp. 177-178: 10-13; pp. 182-197: 2, 7, 13, 17, 20, 25, 37, 39, 41-45; pp. 239-243: 1, 8; pp. 243-248: 1-8

Problems:

1. The dice game of craps (see also, Ross, p. 58) begins with the player (called the shooter) rolling two fair dice. If the result is a 2, or 3, or 12, the shooter loses, while if the result is a 7 or 11, the shooter wins.

(a) What is the probability that the shooter loses on the first roll? What is the probability that the shooter wins on the first roll?

(b) If the sum of the dice on the first roll is any of 4, 5, 6, 8, 9, 10, that number is called the shooter's point. For each number i in the set {4, 5, 6, 8, 9, 10}, find the probability that the shooter's point is i. I need six answers here, folks!

(c) Suppose that the shooter's point is i where i is some number in {4, 5, 6, 8, 9, 10}. The shooter now rolls the two dice again. If the result is a 7, the shooter loses (this is referred to as crapping out). If the result is i, the shooter wins (this is referred to as making the point). If the result is neither i nor 7, the shooter rolls again. This process continues until the shooter either makes the point or craps out. Given that the shooter's point is i, what is the conditional probability that the shooter makes the point? Naturally, the answer will depend on i, so here too, I need six answers.

(d) Use the above results to compute the probability of winning at craps.

(e) Given that the shooter's point is 8, what is the probability that the shooter makes it "the hard way," that is, by rolling two fours? Generally, bets are offered at 10-to-1 odds that the shooter does not make the point 8 the hard way. That is, if you bet $1, you win $10 (plus your $1 back!) if the shooter makes 8 the hard way, and you lose the $1 that you bet if the shooter craps out or makes 8 by rolling 2-6, 3-5, 5-3, or 6-2). In the long run over many such bets, do you expect to make money or lose money or come out even?

2. The probability that you can hear the sound of a pin dropping onto a table during a long-distance telephone call from Champaign-Urbana to Hollywood is p_0 if the call is being carried over the AT&T network and p_1 if the call is being carried over the Sprint network. Assume that both p_0 and p_1 are quite small and that $p_1 > p_0$. You call Candice Bergen from a payphone owned by Sleazo Telecom Corporation which happens to lease its long-distance lines either from AT&T or Sprint (but you don't know which!), and she agrees to drop pins one by one onto a table until you hear the sound of one dropping. Suppose that you hear her counting "One, two, three, … " as she drops the pins, but you hear nothing else until she says "thirtyfour" and you finally hear the sound made by the pin dropping onto the table.

(a) Let H_0 and H_1 denote the hypotheses that the call is being carried by AT&T and Sprint respectively. What is the likelihood ratio LR for the observation that the 34th pin dropped was the first one heard?

(b) The maximum-likelihood decision compares LR to the threshold 1 and announces in favor of H_0 and H_1 according as LR < 1 or LR > 1 . Show that this decision rule can be expressed in terms of a threshold test on k, the number of the first pin that you heard being dropped.

(c) If $p_1 = 0.04$ and $p_0 = 0.02$, what is the maximum-likelihood decision if the 34th pin is the first one heard?

(d) AT&T is the lessor of 95% of all long-distance telephone lines while Sprint is the lessor of the remaining 5%, and thus it is reasonable to assume that $P(H_0) = 0.95$. What is the Bayesian decision if the 34th pin is the first one heard?

(e) Noncredit exercise: Whom do you think is *my* long-distance carrier?

3. An ARQ communication system uses packets that have 72 bits of overhead including address and header information, CRC bits etc. If the channel bit error probability is 10^{-5}, what is the optimum packet size in bits? Here, as in class, the goal is to maximize the ratio of the number of *data* bits successfully sent over the channel to the total number of bits (including overhead, repeated packet transmissions, etc.) that had to be transmitted over the channel to achieve this goal.

4. Let the random variable I_D denote the indicator function of an event D, that is,

$$I_D(\omega) = \begin{cases} 1 & \text{if } \omega \quad D, \\ 0 & \text{if } \omega \quad D. \end{cases}$$

Let A, B, and C denote independent events with probability 1/2, and define the random variable X by $X(\omega) = I_A(\omega) + 2I_B(\omega) - I_C(\omega)$.

(a) What are the values taken on by the random variable **X** ?

(b) Find the cumulative probability distribution function $F_{\mathbf{X}}(u)$ and the probability mass function $p_{\mathbf{X}}(u)$ of the random variable **X**. Be very careful in specifying the values of $F_{\mathbf{X}}(u)$ at points where the function is discontinuous.

5. Which of the following are valid cumulative probability distribution functions (cdfs) ? For those that are not valid cdfs, state at least one property of cdfs which is not satisfied. For those which are valid cdfs, compute $P\{|\mathbf{X}| > 0.5\}$.

(a) $$F_{\mathbf{X}}(u) = \begin{cases} 0, & u < 1, \\ 2u - u^2, & 1 \le u \le 2, \\ 1, & u > 2. \end{cases}$$

(b) $$F_{\mathbf{X}}(u) = \begin{cases} (1/2)\exp(2u), & u < 0, \\ 1 - (1/4)\exp(-3u), & u \ge 0. \end{cases}$$

(c) $$F_{\mathbf{X}}(u) = \begin{cases} (1/2)\exp(2u), & u \le 0, \\ 1 - (1/4)\exp(-3u), & u > 0. \end{cases}$$