Homework 9 Solutions

Problem 1:

8.10. First, let us adopt some notation. Let NW and NL denote "Napoleon wins" and "Napoleon loses," respectively. Also, let "P&E" denote that the Prussians and English have joined forces.

The best way to handle this problem is to express Bayes' theorem in odds form. Show that

$$\frac{P(NW \mid P\&E)}{P(NL \mid P\&E)} = \frac{P(P\&E \mid NW)}{P(P\&E \mid NL)} \frac{P(NW)}{P(NL)}.$$

We have that P(NW) = 0.90 and P(NW | P&E) = 0.60, and so P(NL) = 0.10 and P(NL | P&E) = 0.40. Thus, we can substitute:

$$\frac{0.60}{0.40} = \frac{P(P\&E \mid NW)}{P(P\&E \mid NL)} \frac{0.90}{0.10}$$

or

$$\frac{P(P\&E \mid NW)}{P(P\&E \mid NL)} = \frac{1}{6}.$$

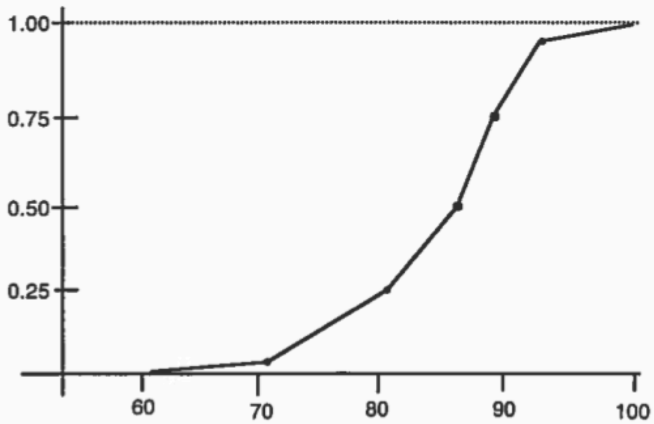
Thus, Napoleon would have had to judge the probability of the Prussian and English joining forces as six time more likely if he is to lose than if he is to win.

Problem 2:

8.11. This problem requires a student to assess a subjective CDF for his or her score in the decision analysis course. Thus, answers will vary considerably, depending on personal assessments. For illustrative purposes, assume that the student assesses the 0.05, 0.25, 0.50, 0.75, and 0.95 fractiles:

$x_{0.05} = 71$ $\quad$ $x_{0.25} = 81$ $\quad$ $x_{0.50} = 0.87$ $\quad$ $x_{0.75} = 89$ $\quad$ $x_{0.95} = 93$

These assessments can be used to create a subjective CDF:



To use these judgments in deciding whether to drop the course, we can use either bracket medians or the Pearson-Tukey method. The Pearson-Tukey method approximates the expected DA score as:

$$E_{P\text{-}T}(\text{DA Score}) = 0.185\ (71) + 0.63\ (87) + 0.185\ (93) = 85.15.$$

Assume that the student has a GPA of 2.7. Using $E_{P\text{-}T}$(DA Score) = 85.15 to calculate expected salary,

$$E(\text{Salary} \mid \text{Drop Course}) = \$4000\ (2.7) + \$16{,}000 = \$26{,}800$$

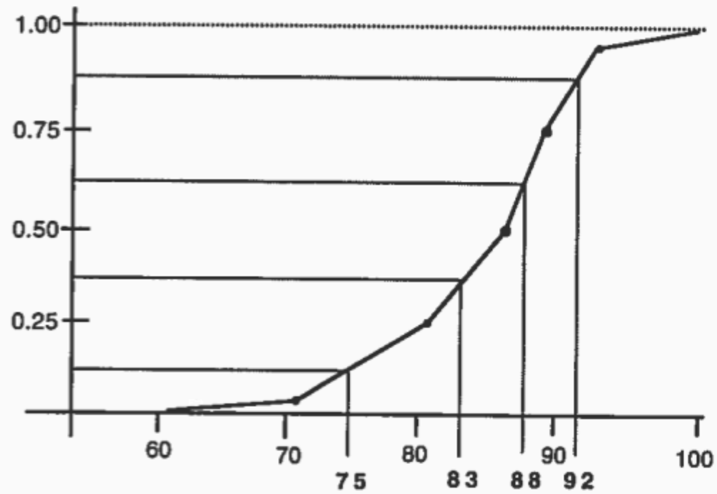
Assume that the student has a GPA of 2.7. Using $E_{P\text{-}T}$(DA Score) = 85.15 to calculate expected salary,

$$E(\text{Salary} \mid \text{Drop Course}) = \$4000\ (2.7) + \$16{,}000 = \$26{,}800$$

$$E(\text{Salary} \mid \text{Don't drop}) = 0.6\ (\$4000 \times 2.7) + 0.4\ (\$170 \times E_{P\text{-}T}(\text{DA Score})) + \$16{,}000$$
$$= 0.6\ (\$4000 \times 2.7) + 0.4\ (\$170 \times 85.15) + \$16{,}000$$
$$= \$28{,}270.$$

Thus, the optimal choice is not to drop the course.

To use the bracket median approach, we first determine that bracket medians for four equal-probability intervals would be approximately 75, 83, 88, and 92.



Thus, the bracket-median approximation would be

$E_{BM}(\text{DA Score}) \approx 0.25\,(75) + 0.25\,(83) + 0.25\,(88) + 0.25\,(92) = 84.5$

Using this in calculating expected salaries:

$E(\text{Salary} \mid \text{Drop Course}) = \$4000\,(2.7) + \$16{,}000 = \$26{,}800$

$$
\begin{aligned}
E(\text{Salary} \mid \text{Don't drop}) &= 0.6\,(\$4000 \times 2.7) + 0.4\,(\$170 \times E_{BM}(\text{DA Score})) + \$16{,}000 \\
&= 0.6\,(\$4000 \times 2.7) + 0.4\,(\$170 \times 84.5) + \$16{,}000 \\
&= \$28{,}226.
\end{aligned}
$$

Again, the conclusion is not to drop the course.

Problem 3:

a. For each project, the investor appears to believe that

$$
\begin{aligned}
E(\text{profit}) &= 0.5\,(150{,}000) + 0.5\,(-100{,}000) \\
&= 75{,}000 - 50{,}000 \\
&= 25{,}000
\end{aligned}
$$

b. However, since only one of the projects will succeed, he will gain $150,000 for the successful project, but lose $100,000 for each of the other two. Thus, he is guaranteed to lose $50,000 no matter what happens.

c. For a set of mutually exclusive and collectively exhaustive outcomes, he appears to have assessed probabilities that add up to 1.5.

Problem 4:

a. If he will accept Bet 1, then it must have non-negative expected value:

$$P(\text{Cubs win}) (\$20) + [1 - P(\text{Cubs win})] (-\$30) \geq 0.$$

Increasing the "win" amount to something more than \$20, or decreasing the amount he must pay if he loses (\$30) will increase the EMV of the bet. However, reducing the "win" amount or increasing the "lose" amount may result in a negative EMV, in which case he would not bet. The same argument holds true for Bet 2.

b. Because he is willing to accept Bet 1, we know that

$$P(\text{Cubs win}) (\$20) + [1 - P(\text{Cubs win})] (-\$30) \geq 0,$$

which can be reduced algebraically to

$$P(\text{Cubs win}) \geq 0.60.$$

Likewise, for Bet 2, we know that

$$P(\text{Cubs win}) (-20) + [1 - P(\text{Cubs win})] (\$40) \geq 0.$$

This can be reduced to $P(\text{Cubs win}) \leq 0.67$. Thus, we have $0.60 \leq P(\text{Cubs win}) \leq 0.67$.

c. Set up a pair of bets using the strategy from Chapter 8. From Bet 1 we infer that P(Cubs win) = 0.60, and from Bet 2 P(Yankees win) = 0.33. Use these to make up Bets A and B:

A: He wins 0.4 X if Cubs win
He loses 0.6 X if Yankees win

B: He wins 0.67 Y if the Yankees win
He loses 0.33 Y if the Cubs win

We can easily verify that the EMV of each bet is equal to 0:

$$\begin{aligned} EMV(A) &= P(\text{Cubs win})\,(0.4\,X) + [1 - P(\text{Cubs win})]\,(-0.6\,X) \\ &= 0.6\,(0.4\,X) - 0.4\,(0.6\,X) \\ &= 0 \end{aligned}$$

$$\begin{aligned} EMV(B) &= P(\text{Yankees win})\,(0.67\,Y) + [1 - P(\text{Yankees win})]\,(-0.33\,Y) \\ &= 0.33\,(0.67\,Y) - 0.67\,(0.33\,Y) \\ &= 0 \end{aligned}$$

If the Cubs win, his position is:

$$0.4\,X - 0.33\,Y = W$$

If the Yankees win:

$$-0.6\,X + 0.67\,Y = Z$$

Following the strategy in the book, set W = Z = -$100 to be sure that he pays us $100 net, regardless of what happens:

$$\begin{aligned} 0.4\,X - 0.33\,Y &= -\$100 \\ -0.6\,X + 0.67\,Y &= -\$100 \end{aligned}$$

Now solve these two equations for X and Y to obtain X = Y = -$1500. Thus, the original bets A and B become:

A: He wins -$600 if Cubs win
He loses -$900 if Yankees win

B: He wins -$1000 if the Yankees win
He loses -$500 if the Cubs win

The minus sign means that he is taking the "other side" of the bet, though (i.e. winning -$600 is the same as losing $600). Thus, these two bets really are:

A: He loses $600 if Cubs win
He wins $900 if Yankees win

B: He loses $1000 if the Yankees win
He wins $500 if the Cubs win

Finally, compare these bets to Bets 1 and 2 in the book. He said he would bet on the Cubs at odds of 3:2 or better, but we have him betting on the Cubs (in Bet B) at odds of 2:1, which is worse. (That is, he has to put up 2 dollars to win 1, rather than 1.5 to win 1.) The same problem exists with bet A: he is betting on the Yankees at odds of 2:3, which is worse than 1:2. As a result, he will not accept either of these bets!

The reason for this result is that the solutions for X and Y are negative. In fact, it is possible to show algebraically that if A and B are both negative, then X and Y will both be negative.

This has the effect of reversing the bets in such a way that your friend will accept neither. The conclusion is that, even though his probabilities appear incoherent, you cannot set up a Dutch book against him.

Problem 5:

8.27. a, b. Most people choose A and D because these two are the options for which the probability of winning is known.

c. Choosing A and D may appear to be consistent because both of these involve known probabilities. However, consider the EMVs for the lotteries and the implied values for P(Blue). If A is preferred to B, then

$$\text{EMV(A)} > \text{EMV(B)}$$

$$\frac{1}{3}(1000) > \text{P(Blue)}\,(1000)$$

$$\text{P(Blue)} < \frac{1}{3}.$$

However, if D is preferred to C, then

$$\text{EMV(D)} > \text{EMV(C)}$$

$$\text{P(Blue)}\,(1000) + \text{P(Yellow)}\,(1000) > \frac{1}{3}(1000) + \text{P(Yellow)}\,(1000)$$

$$\text{P(Blue)} > \frac{1}{3}.$$

The inconsistency arises because it clearly is not possible to have both $\text{P(Blue)} < \frac{1}{3}$ and $\text{P(Blue)} > \frac{1}{3}$. (Exactly the same result obtains if we use the utility of \$1000, instead of the dollar value.)