ECE 307

Homework 7 Solutions

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10.12: PROBLEM FORMULATION

- **This is a multi–period planning problem with a 7–*month* horizon**
- **Define the following for a backward recursion**
  - **stage: a month of the planning period**
  - **state variable: the number of crankcases S_n left over from the stage $(n-1)$, $n = 1, 2, \ldots, 7$ with $S_7 = 0$ and S_0 unspecified**

10.12: PROBLEM FORMULATION

- **decision variables: purchase amount d_n for stage n, $n = 1, 2, \ldots, 7$**
- **transition function: the relationship between the amount in inventory, purchase decision and demand in stages n and $(n-1)$**

$$S_{n-1} = S_n + d_n - D_n \qquad\qquad n = 1, 2, \ldots, 7$$

where,

$$D_n = \text{demand at stage } n, \; n = 1, 2, \ldots, 7$$

10.12: PROBLEM FORMULATION

- return function: costs of purchase in stage n plus the inventory holding costs given by the mathematical expression

$$f^*_n(S_n) = C_n + (S_n + d_n - D_n)\,0.50 + f^*_{n-1}(S_{n-1})$$

costs of lot size ordered

per unit inventory charges

and

$$f^*_0(S_0) = 0$$

10.12: STAGE 1 SOLUTION

$$D_1 = 600$$

$$f_1^*(S_1) = \min_{d_1}\left\{C_1 + (S_1 + d_1 - D_1)0.50\right\}$$

S_1	*value of f_1 for d_1*				$f_1^*(S_1)$	d_1^*
	0	500	1,000	1,500		
0			5,200	7,950	5,200	1,000
100		3,000	5,250	8,000	3,000	500
200		3,050	5,300	8,050	3,050	500
300		3,100	5,350	8,100	3,100	500
400		3,150	5,400	8,150	3,150	500
500		3,200	5,450	8,200	3,200	500
600	*0*	3,250	5,500	8,250	*0*	*0*

10.12: STAGE 2 SOLUTION

$$D_2 = 1{,}200$$

$$f_2^*(S_2) = \min_{d_2}\left\{C_2 + (S_2 + d_2 - D_2)0.50 + f_1^*(S_2 + d_2 - D_2)\right\}$$

S_2	*value of f_2 for d_2*				$f_2^*(S_2)$	d_2^*
	0	500	1,000	1,500		
0				10,750	10,750	1,500
100				10,850	10,850	1,500
200			10,200	10,950	10,200	1,000
300			8,050	7,800	7,800	1,500
400			8,150		8,150	1,000
500			8,250		8,250	1,000
600			8,350		8,350	1,000

10.12: STAGE 3 SOLUTION

$$D_3 = 900$$

$$f_3^*(S_3) = \min_{d_3}\left\{C_3 + (S_3 + d_3 - D_3)0.50 + f_2^*(S_3 + d_3 - D_3)\right\}$$

S_3	value of f_3 for d_3				$f_3^*(S_3)$	d_3^*
	0	500	1,000	1,500		
0			15,900	16150	15,900	1,000
100			15,300		15,300	1,000
200			12,950		12,950	1,000
300			13,350		13,350	1,000
400		13,750	13,500		13,500	1,000
500		13,900	13,650		13,650	1,000
600		13,300			13,300	500

10.12: STAGE 4 SOLUTION

$D_4 = 400$

$$f_4^*(S_4) = \min_{d_4}\left\{C_4 + (S_4 + d_4 - D_4)0.50 + f_3^*(S_4 + d_4 - D_4)\right\}$$

S_4	*value of f_4 for d_4*				$f_4^*(s_4)$	d_4^*
	0	500	1,000	1,500		
0		18,350	18,600		18,350	500
100		16,050			16,050	500
200		16,500			16,500	500
300		16,700			16,700	500
400	15,900	16,900			15,900	*0*
500	15,350	16,600			15,350	*0*
600	13,050				13,050	*0*

10.12: STAGE 5 SOLUTION

$$D_5 = 800$$

$$f_5^*(S_5) = \min_{d_5}\left\{C_5 + (S_5 + d_5 - D_5)0.50 + f_4^*(S_5 + d_5 - D_5)\right\}$$

S_5	*value of f_5 for d_5*				$f_5^*(S_5)$	d_5^*
	0	500	1,000	1,500		
0			21,600		21,600	1,000
100			21,850		21,850	1000
200			21,100		21,100	1,000
300		21,350	20,600		20,600	1,000
400		19,100	18,350		18,350	1,000
500		19,600			19,600	500
600		19,850			19,850	500

10.12: STAGE 6 SOLUTION

$$D_6 = 1{,}100$$

$$f_6^*(S_6) = \min_{d_6}\left\{C_6 + \left(S_6 + d_6 - D_6\right)0.50 + f_5^*\left(S_6 + d_6 - D_6\right)\right\}$$

S_6	value of f_6 for d_6				$f_6^*(S_6)$	d_6^*
	0	500	1,000	1,500		
0				26,050	26,050	1,500
100			26,600	27,350	26,600	1,000
200			26,900	27,650	26,900	1,000
300			26,200		26,200	1,000
400			25,750		25,750	1,000
500			23,550		23,550	1,000
600		24,600	24,850		24,600	500

10.12: STAGE 7 SOLUTION

- For stage 7, $D_7 = 700$ and

$$f_7^*(S_7) = \min_{d_7}\left\{C_7 + \left(S_7 + d_7 - D_7\right)0.50 + f_6^*\left(S_7 + d_7 - D_7\right)\right\}$$

- Optimal total cost over 7 months = $ 31,350

  obtained with the purchasing policy given below

month	1	2	3	4	5	6	7
amount of material	1,000	1,000	1,000	0	1,000	1,500	500

SOLUTION CHARACTERISTICS

- The initial state variable was specified $\left(S_1 = 0 \right)$ and the final output state S_8 was a "choice" variable; thus, this is an *initial value DP* problem
- The state variables and decision variables are decomposed into discrete, mutually exclusive sets at each stage of the *DP* solution
- Since each input state variable gives rise to only a single output state variable, this problem is known as a *serial DP* problem

10.14 (*a*): PROBLEM FORMULATION

- The problem is a *transportation problem* which is a special case *LP*

$$min\, Z \;=\; min \sum_{i=1}^{4} \sum_{j=1}^{6} c_{ij} x_{ij}$$

s.t.

$$\sum_{j=1}^{6} x_{ij} \;=\; 1 \qquad \forall\, i = 1, \ldots, 4$$

$$\sum_{i=1}^{4} x_{ij} \;\leq\; 1 \qquad \forall\, j = 1, \ldots, 6$$

$$x_{ij} \in \{0, 1\}$$

10.14 (*b*): *DP* SOLUTION

- **Define the following:**
  - **stage: car numbered** $n = 1, 2, 3, 4$
  - **state variable** $\underline{s}_n$ **: vector whose dimension is the number of unassigned markets with each component given by the number of the unassigned market**

10.14 (*b*): *DP* SOLUTION

- **decision variable: unassigned market d_n, a component of $\underline{s}_n$, with $1 \leq d_n \leq 6,\ n = 1, \ldots, 4$**
- **stage n costs: costs $r_n\left(d_n\right)$ of the assignment of the car n to the market d_n**
- **return function: total costs at stage n**

$$f_n^*\left(\underline{s}_n\right) = \min_{d_n}\left\{r_n\left(d_n\right) + f_{n-1}^*\left(\underline{s}_{n-1}\right)\right\}$$

with

10.14 (b): DP SOLUTION

d_n is a component of $\underline{s}_n$

$\underline{s}_{n-1}$ is the reduced vector obtained from $\underline{s}_n$ via the removal of d_n

- objective:

$$min\, Z = \sum_{n=1}^{4} r_n\left(d_n\right), \ d_n \text{ is a component of } \underline{s}_n, n = 1, 4$$

- transition relationship: $\underline{s}_{n-1}$ is the reduced vector obtained from $\underline{s}_n$ by the removal of the component d_n

10.14 (*b*): STAGE 1 SOLUTION

- In stage 1, we allocate car 1, having already allocated 3 markets to the other 3 cars
- Consequently, there are

$$\frac{6!}{3!3!} = 20$$

possible states $\underline{s}_1$ for which to make a decision

10.14 (*b*): STAGE 1 SOLUTION

state number	$\underline{s}_1$	value of f_1 for decision d_1						d_1^*	f_1^*
		1	2	3	4	5	6		
1	[1,2,3]	7	12	9				1	7
2	[1,2,4]	7	12		15			1	7
3	[1,2,5]	7	12			8		1	7
4	[1,2,6]	7	12				14	1	7
5	[1,3,4]	7		9	15			1	7
6	[1,3,5]	7		9		8		1	7
7	[1,3,6]	7		9			14	1	7
8	[1,4,5]	7			15	8		1	7
9	[1,4,6]	7			15		14	1	7
10	[1,5,6]	7				8	14	1	7
11	[2,3,4]		12	9	15			3	9
12	[2,3,5]		12	9		8		5	8
13	[2,3,6]		12	9			14	3	9
14	[2,4,5]		12		15	8		5	8
15	[2,4,6]		12		15		14	2	12
16	[2,5,6]		12			8	14	5	8
17	[3,4,5]			9	15	8		5	8
18	[3,4,6]			9	15		14	3	9
19	[3,5,6]			9		8	14	5	8
20	[4,5,6]				15	8	14	5	8

10.14 (b): STAGE 2 SOLUTION

- In stage 2, we assign car 2 having already assigned cars 4 and 3 to two of the six markets
- The number of possible states $\underline{s}_2$ is

$$\frac{6!}{2!4!} = 15$$

- For each state $\underline{s}_2$, we compute

$$f_2^*\left(\underline{s}_2\right) = \min_{d_2}\left\{r_2\left(d_2\right) + f_1^*\left(\underline{s}_1\right)\right\},$$

with d_2 as one of the components of $\underline{s}_2$ and $\underline{s}_1$ as the reduced vector without the d_1 component

10.14 (*b*): STAGE 2 SOLUTION

state number	$\underline{s}_2$	*value of f_2 for decision d_2*						d_2^*	f_2^*
		1	2	3	4	5	6		
1	[1, 2, 3, 4]	14	17	12	19			3	12
2	[1, 2, 3, 5]	13	17	12		13		3	12
3	[1, 2, 3, 6]	14	17	12			20	3	12
4	[1, 2, 4, 5]	13	17		19	13		1, 5	13
5	[1, 2, 4, 6]	17	17		19		20	1, 2	17
6	[1, 2, 5, 6]	13	17			13	20	1, 5	13
7	[1, 3, 4, 5]	13		12	19	13		3	12
8	[1, 3, 4, 6]	14		12	19		20	3	12
9	[1, 3, 5, 6]	13		12		13	20	3	12
10	[1, 4, 5, 6]	13			19	13	20	1, 5	13
11	[2, 3, 4, 5]		18	13	20	15		3	13
12	[2, 3, 4, 6]		19	17	21		22	3	17
13	[2, 3, 5, 6]		18	13		15	21	3	13
14	[2, 4, 5, 6]		18		20	18	21	2, 5	18
15	[3, 4, 5, 6]			13	20	15	21	3	13

10.14 (b): STAGE 3 SOLUTION

- In stage 3, we assign car 3 having already assigned car 4 to one of the six markets
- The number of possible states in stage 3 is

$$\frac{6!}{5!1!} = 6$$

- For each state $\underline{s}_3$, we compute

$$f_3^*\left(\underline{s}_3\right) = \min_{d_3}\left\{r_3\left(d_3\right) + f_2^*\left(\underline{s}_2\right)\right\},$$

with d_3 as a component of $\underline{s}_3$ and $\underline{s}_2$ as the reduced vector without the component d_3

10.14 (*b*)

state number	$\underline{s}_3$	*value of f_3 for decision d_3*						d_3^*	f_3^*
		1	2	3	4	5	6		
1	[1, 2, 3, 4, 5]	21	22	20	28	19		5	19
2	[1, 2, 3, 4, 6]	25	22	24	28		24	2	22
3	[1, 2, 3, 5, 6]	21	22	30		19	24	5	19
4	[1, 2, 4, 5, 6]	26	23		29	24	25	2	23
5	[1, 3, 4, 5, 6]	21		20	28	19	24	5	19
6	[2, 3, 4, 5, 6]		23	25	29	24	25	2	23

10.14 (*b*): STAGE 4 SOLUTION

- In stage 4, car 4 is assigned to the market with the lowest return for all markets
- There is a single state $\underline{s}_1 = [1, 2, 3, 4, 5, 6]$, for which the optimal decision d_4^* is determined

s_4	*value of f_4 for decision d_4*						d_4^*	f_4^*
	1	2	3	4	5	6		
[1, 2, 3, 4, 5, 6]	32	30	31	33	29	30	5	29

10.14 (*b*): THE OPTIMAL SOLUTION

car	*market*	*cost*
4	5	7
3	4	10
2	3	5
1	1	7
total costs		29