Homework 5 Solutions:

1) Solve using the Hungarian Method:

bids house	painter 1	2	3	4	
1	2.5	1.3	3.6	1.8	−1.3
2	2.9	1.4	5.0	2.2	−1.4
3	2.2	1.6	3.2	2.4	−1.6
4	3.1	1.8	4	2.5	−1.8

subtract 1.3 from each element in this row

↓

bids house	painter 1	2	3	4
1	1.2	0	2.3	0.5
2	1.5	0	3.6	0.8
3	0.6	0	1.6	0.8
4	1.3	0	2.2	0.7
	−0.6		−1.6	−0.5

subtract 0.5 from each element in this column

↓

bids house	painter 1	2	3	4	
1	0.6	0	0.7	0	
2	0.9	0	2	0.3	−0.3
3	0	0	0	0.3	
4	0.7	0	0.6	0.2	−0.2
		+0.3			

bids house	painter 1	2	3	4	
1	0.6	0.3	0.7	0	−0.3
2	0.6	0	1.7	0	
3	0	0.3	0	0.3	
4	0.5	0.1	0.4	0	−0.1
				+0.3	

bids	painter				
house	1	2	3	4	
1	0.6	0.3	0.7	0	−0.3
2	0.6	0	1.7	0	
3	0	0.3	0	0.3	
4	0.5	0.1	0.4	0	−0.1
				+0.3	

bids	painter				
house	1	2	3	4	
1	0.3	0.3	0.4	0	−0.3
2	0.6	0	1.7	0.3	−0.6
3	0	0.3	0	0.6	
4	0.4	0	0.3	0.3	−0.3
		+0.6		+0.3	

bids	painter			
house	1	2	3	4
1	0			0
2		0		0
3	0		0	
4			0	

Notation:

$x_{ij} = 1 \Rightarrow$ house i gets painted by painter j

$x_{ij} = 0 \Rightarrow$ house i does not get painted by painter j

Optimal solution:

$x_{14} = x_{22} = x_{31} = x_{43} = 1$, all other $x_{ij} = 0$

Problem 2:

$x_{ij} \triangleq$ flow in barrels / day from i to j with

$i,j \in \{F1, F2, A, B, R\}$

$$\min. \quad .10\, x_{F1,A} + .35\, x_{F1,B} + .25\, x_{F2,A} + .56\, x_{F2,B}$$

$$+ .12\, x_{A,B} + .12\, x_{A,B} + .40\, x_{A,R} + .33\, x_{B,R}$$

s.t.

$$x_{F1,A} + x_{F1,B} \le 1500$$

$$x_{F2,A} + x_{F2,B} \le 1200$$

} supply

$$x_{F1,A} + x_{F2,A} + x_{B,A} - x_{A,B} - x_{A,R} = 0$$

$$x_{F1,B} + x_{F2,B} + x_{A,B} - x_{B,A} - x_{B,R} = 0$$

} balance

$$x_{A,R} + x_{B,R} = 2000 \quad \text{refinery demand}$$

$$\text{all } x_{i,j} \ge 0$$

By introducing a dummy market we construct a 2x2 standard transportation problem with data given in the following table:

	***Refinery*1**	***Refinery*2**	***Total***
***Field*1**	1,500	0	1,500
***Field*2**	500	700	700
Total	2,000	700	2,700

Where the cost from Field 1 to Refinery 1 is 0.5 and 0 for Refinery 2. The cost from Field 2 to Refinery 1 is 0.65 and 0 to Refinery 2.

Then by inspection:

$x_1^* = 1{,}500$

$x_2^* = 500$

all processed through Axel

Problem 3:

The formulation of the problem into a transportation problem is given in the table below; with the entry c_{ij} in the matrix representing the costs from node i to node j.

Node	1	2	3	4	5	6	*total supply*
1	∞	2	1	∞	∞	∞	30
2	∞	∞	3	1	3	∞	20
3	∞	∞	∞	2	5	∞	35
4	∞	∞	∞	∞	∞	2	18
5	∞	∞	∞	6	∞	3	22
6	∞	∞	∞	∞	∞	∞	0
total demand	0	17	30	30	25	30	

One basic feasible solution could be:

$$x_{12} = 8, x_{13} = 22, x_{23} = 0, x_{24} = 3, x_{25} = 5,$$
$$x_{34} = 15, x_{35} = 7, x_{56} = 12, x_{46} = 18$$

total cost: \$193

Problem 4:

(a), (b) and (c)

	$v_1=7$	$v_2=6$	$v_3=2$	$v_4=5$	
$u_1=0$	(20) 7	(30)$-\theta$ 6	5	$+\theta$ 4	50
$u_2=1$	9	(10)$+\theta$ 7	(30) 3	(10)$-\theta$ 6	50
$u_3=-2$	8	8	7	(50) 3	50
	20	40	30	60	

$\bar{c}_{14} < 0$. x_{14} enters. x_{24} leaves. Max $\theta = 10$.

	$v_1=7$	$v_2=6$	$v_3=2$	$v_4=4$
$u_1=0$	(20) 7	(20) 6	5	(10) 4
$u_2=1$	9	(20) 7	(30) 3	6
$u_3=-1$	8	8	7	(50) 3
	20	40	30	60

Optimal solution; Min Z = \$680.

The new objective function becomes:

$$\text{Min } Z = (\text{old } Z) + 1\,(x_{11} + x_{21} + x_{31}) = (\text{old } Z) + 1\,(20)$$

Thus this change is equivalent to adding a constant to the objective function. Hence optimal solution will not change but the least cost of bussing will increase by \$20.

Problem 5:

a. Let Machines <—> Days
Jobs <—> Courses

$X_{ij} = 1$ if course j is assigned to Day i.

Adding a dummy course with zero costs, we get a standard assignment problem with 5 machines and 5 jobs.

b. Subtract the smallest element in each column from all elements in that column gives the following:

	J_1	J_2	J_3	J_4	J_5
I_1	40	20	50	0	0
I_2	30	10	30	10	0
I_3	50	0	20	0	0
I_4	20	10	10	10	0
I_5	0	0	0	10	0

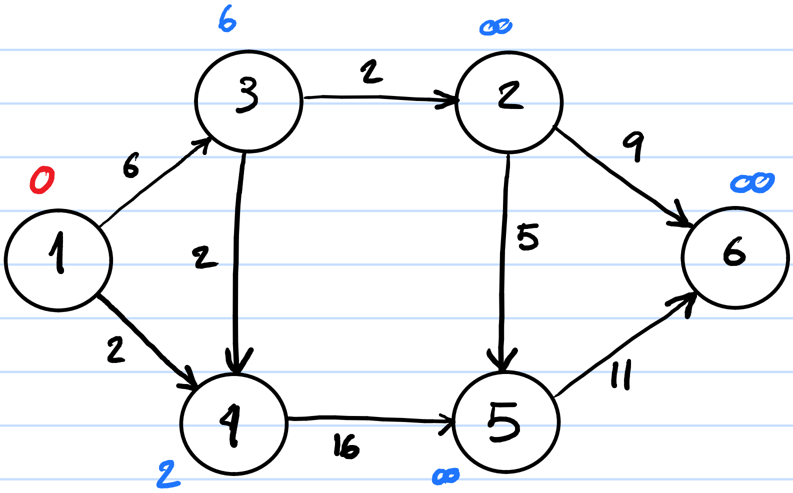
Only 4 assignments are possible with zero cells. Cover all zeros with 4 lines. The smallest element 10 from the uncovered cells is subtracted to give the next table.

	J_1	J_2	J_3	J_4	J_5
I_1	40	20	50	(0)	10
I_2	20	0	20	0	(0)
I_3	50	(0)	20	0	10
I_4	10	0	(0)	0	0
I_5	(0)	0	0	10	10

The optimal assignment:

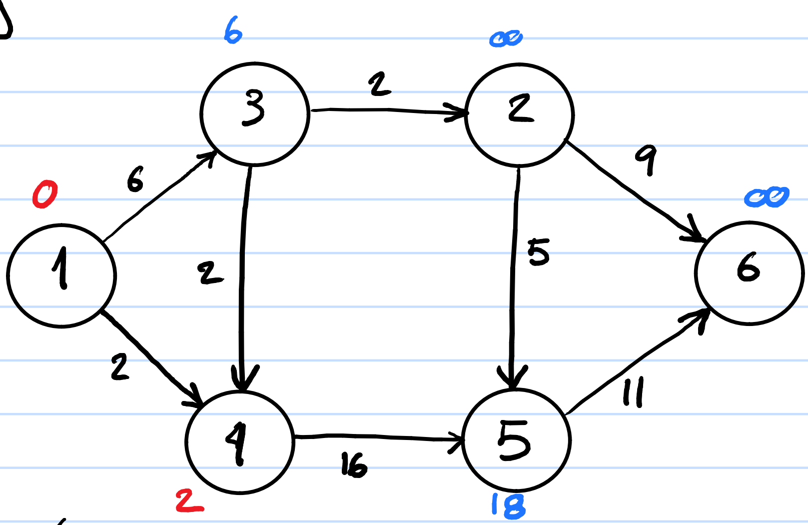
Monday <-> Bioengineering
Wednesday <-> Energy
Thursday <-> Transportation
Friday <-> Ecology

6)



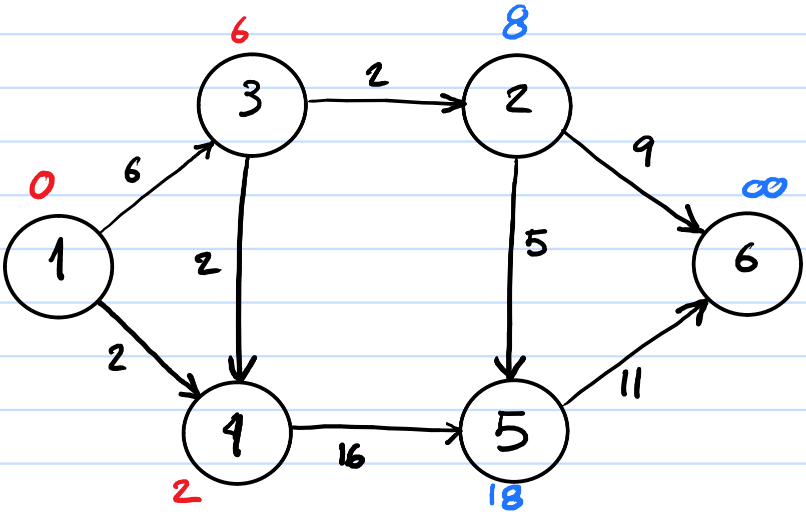
6
∞
0
3
2
2
9
∞
6
1
2
5
6
2
11
4
16
5
2
∞

4



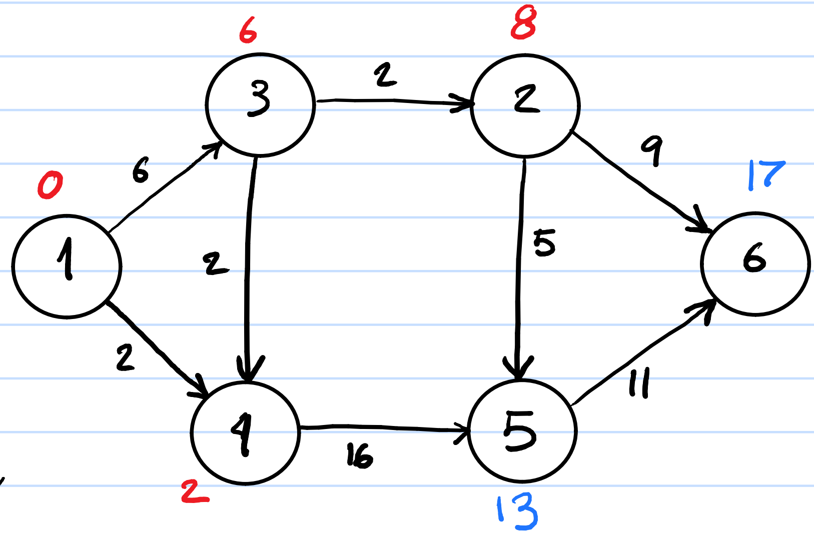
6
∞
3
2
2
9
0
∞
6
1
2
5
6
2
11
4
16
5
2
18

3



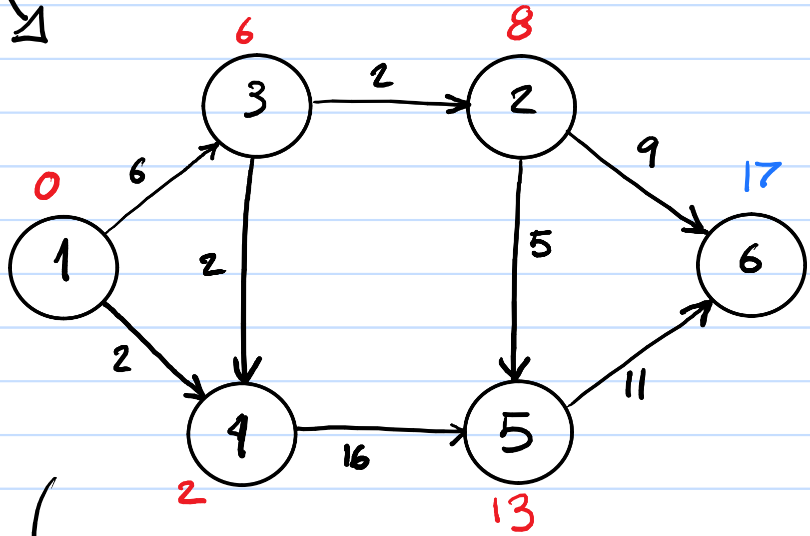
6
8
3
2
2
9
0
∞
6
1
2
5
6
2
11
4
16
5
2
18

2



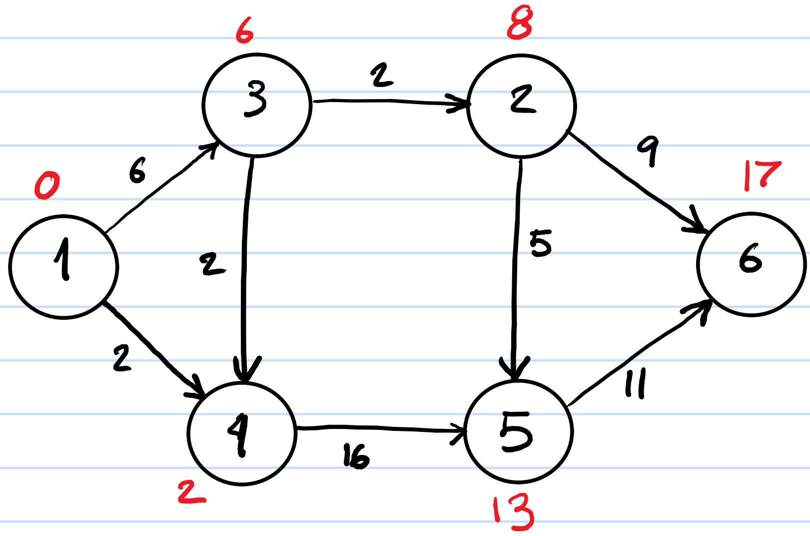
0
1
6
6
3
2
8
2
9
17
6
2
5
2
2
4
16
5
11
13

5



0
1
6
6
3
2
8
2
9
17
6
2
5
2
2
4
16
5
11
13

6



0
1
6
6
3
2
8
2
9
17
6
2
5
2
2
4
16
5
11
13

Problem 7:

$$L[0]=[0,6,\infty,\infty,\infty,11,\infty,\infty,\infty,\infty]$$
$$L[1]=[0,6,\infty,\infty,\infty,11,12,\infty,\infty,\infty]$$
$$L[2]=[0,6,\infty,\infty,22,11,12,\infty,\infty,\infty]$$
$$L[3]=[0,6,42,30,20,11,12,\infty,\infty,\infty]$$
$$L[4]=[0,6,42,30,20,11,12,34,\infty,\infty]$$
$$L[5]=[0,6,42,30,20,11,12,34,57,59]$$

Optimal paths are:

$1-2$, ***length 6***

$1-2-7-3$, ***length 42***

$1-2-7-4$, ***length 30***

$1-2-7-5$, ***length 20***

$1-6$, ***length 11***

$1-2-7$, ***length 12***

$1-2-7-5-8$, ***length 34***

$1-2-7-5-8-9$, ***length 57***

$1-2-7-5-8-10$, ***length 59***

Problem 8

(a) (i) Path : (s) → (2) → (4) → (n)
(ii) Cut : $S = \{s, 1, 2, 3, 4\}$, $\bar{S} = \{n\}$
(iii) $K(S,\bar{S}) = 10 + 20 + 5 = 35$

(b) The following augmenting paths are:

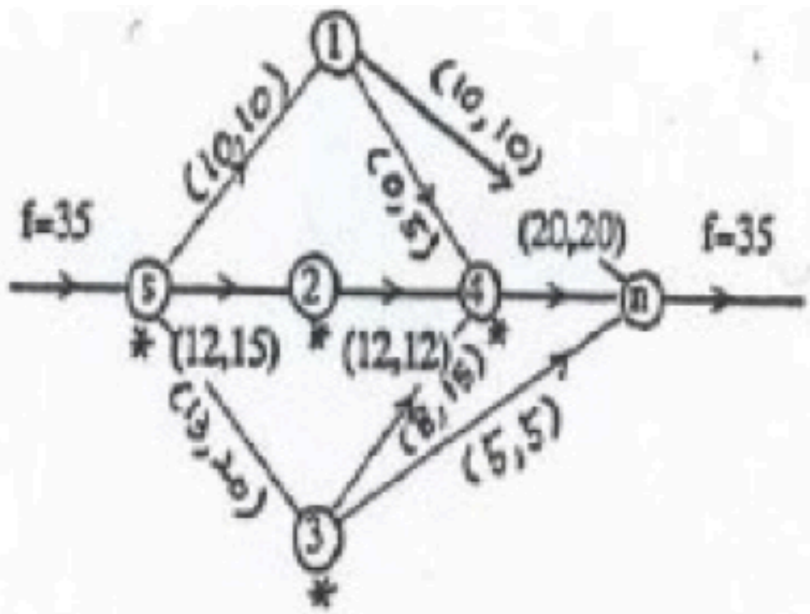
$\delta=10$ (10,10) (10,10) $\delta=10$
→ (s) → (1) → (n) →
$f = 10$

$\delta=12$ (12,15) (12,12) (12,20) $\delta=12$
→ (s) → (2) → (4) → (n) →
$f = 10 + 12 = 22$

$\delta=8$ (8,20) (8,15) (20,20) $\delta=8$
→ (s) → (3) → (4) → (n) →
$f = 22 + 8 = 30$

$\delta=5$ (15,20) (5,5) $\delta=5$
→ (s) → (3) → (n) →
$f = 30 + 5 = 35$

No more flow augmenting paths are possible. Max flow = 35. The flow distribution is given below:



(c)

A minimal cut : S = set of all labeled nodes

= { s,2,3,4 }

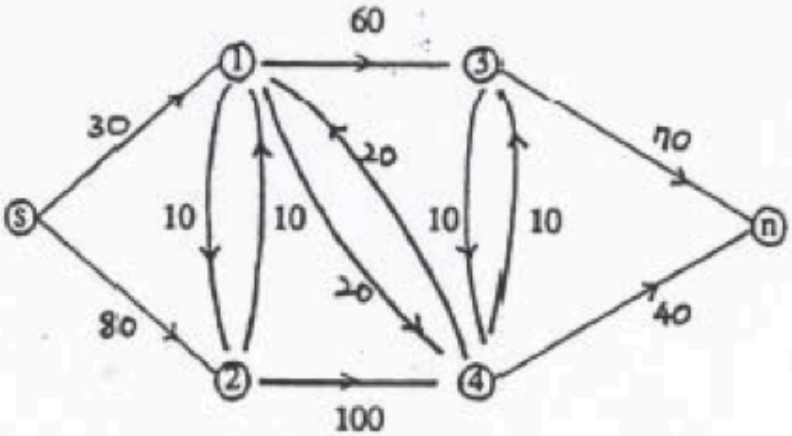
$\bar{S}$ = unlabeled nodes

= { 1, n }

$K(S,\bar{S}) = 10 + 20 + 5 = 35$ = max flow.

Problem 9

Replace the undirected arcs (1,2) ; (1,4) and (3,4) by a pair of directed arcs each.



Applying the max. flow algorithm, the flow augmenting paths are:

δ=30 (30,30) (30,60) (30,70) δ=30
→ S → 1 → 3 → n →
f = 30

δ=40 (40,80) (40,100) (40,400) δ=40
→ S → 2 → 4 → n →
f = 30 + 40 = 70

δ=10 (50,80) (10,10) (40,60) (40,70) δ=10
→ S → 2 → 1 → 3 → n →
f = 70 + 10 = 80

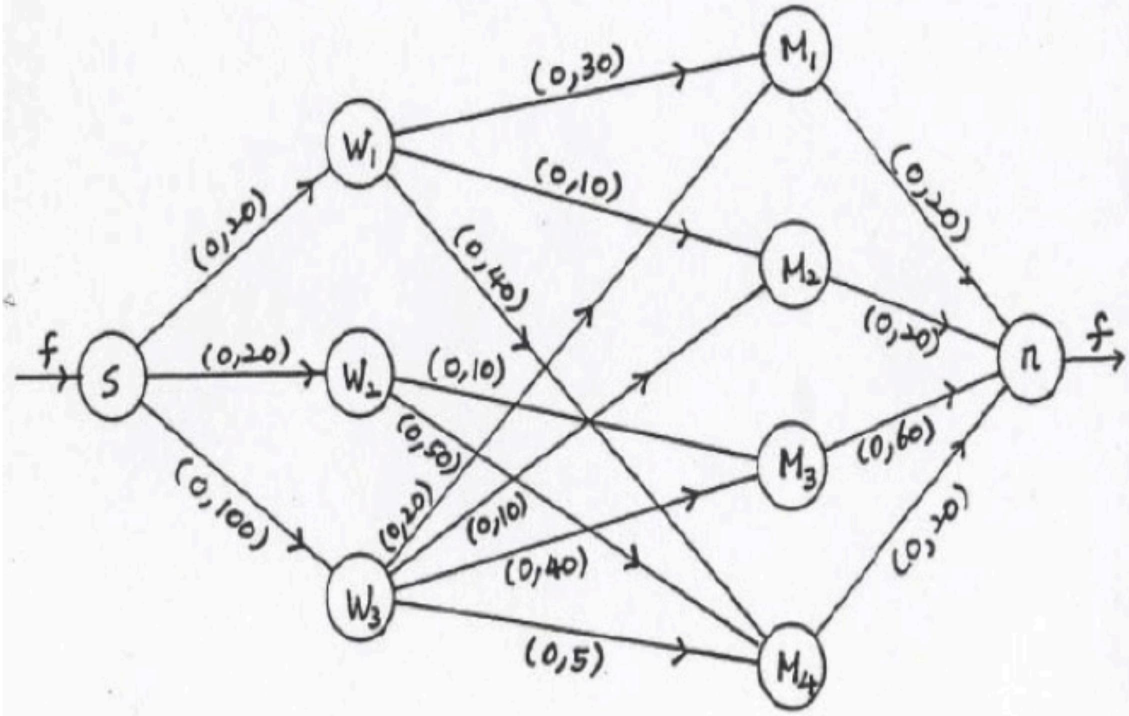
δ=10 (60,80) (50,100) (10,10) (50,70) δ=10
→ S → 2 → 4 → 3 → n →
f = 80 + 10 = 90

δ=20 (80,80) (70,100) (20,20) (60,60) (70,70) δ=20
→ S → 2 → 4 → 1 → 3 → n →
f = 90 + 20 = 110

Max flow = 110. Since $f_{12} = 0$ and $f_{21} = 10$, flow in the undirected arc is from 2 to 1. Similarly one way signs be placed from 4 to 1 and 4 to 3.

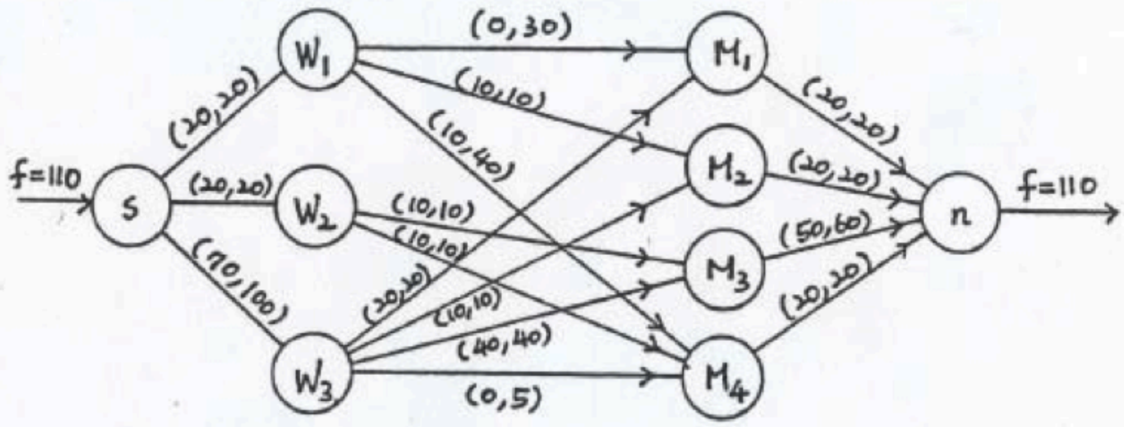
Problem 10

(a)



Find the max flow from s to n in the above network. If max f = sum of all demands = 120, then it is possible to meet the demands with available supply. If max f < 120, then it is not possible. (Note: max f > 120 is not possible since there is a cut S = { s, W_1, W_2, W_3, M_1, M_2, M_3, M_4 }, $\overline{S}$ = {n} whose capacity is 120.)

(b) Applying the max flow algorithm, the maximal flow is 110 < 120. Hence it is not possible to meet all the demands. There will be a shortage off 10 units at M_3. The initial flow distribution is shown below:



Min cut : S = { s, W_1, W_2, W_3, M_1, M_4} ; $\bar{S}$ = {M_2, M_3, n}

Problem 11

	Week 1	Week 2	Week 3	Week 4	Dummy	
Week 1	10	13	16	19	0	700
Week 2 (Normal)	14	10	13	16	0	700
Week 2 (OT)	23	15	18	21	0	200
Week 3 (Normal)	27	19	15	18	0	700
Week 3 (OT)	36	28	20	23	0	200
Week 4	35	27	19	15	0	700
	300	700	900	800	500	

Variables:

x_{1j} = Normal production in week 1 for use in week j for j = 1, 2, 3, 4
x_{2j} = Normal production in week 2 for use in week j for j = 1, 2, 3, 4
x_{3j} = Overtime production in week 2 for use in week j for j = 1, 2, 3, 4
x_{4j} = Normal production in week 3 for use in week j for j = 1, 2, 3, 4
x_{5j} = Overtime production in week 3 for use in week j for j = 1, 2, 3, 4
x_{6j} = Normal production in week 4 for use in week j for j = 1, 2, 3, 4

NOTE: x_{21}, x_{31}, x_{41}, x_{51}, x_{61}, x_{42}, x_{52}, x_{62}, x_{63}, are productions to fill the backorders.

Transportation algorithm is initialized by Least Cost Rule as follows:

	v_1=10	v_2=13	v_3=16	v_4=19	v_5=0	
u_1=0	(300) 10	13	(200) 16	(100) 19	(100) 0	700
u_2=−3	14	(700) 10	(0) 13	16	0	700
u_3=0	23	15	18	21	(200) 0	200
u_4=−1	27	19	(700) 15	18	0	700
u_5=0	36	28	20	23	(200) 0	200
u_6=−1	35	27	19	(700) 15	0	700
	300	700	900	800	500	

The initial basic feasible solution by Least Cost Rule turns out to be optimal. Interpretation (Production schedule)

Produce 600 units on regular time on week 1
Produce 700 units on regular time on week 2
Produce 700 units on regular time on week 3
Produce 700 units on regular time on week 4

No overtime production on any week.