HOMEWORK 4 SOLUTIONS

1)

a)

$$A = \begin{pmatrix} 4 & -2 & 9 & 0 & 0 \\ -1 & 5 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix}$$

$$b = \begin{pmatrix} 22 \\ 1 \\ 5 \end{pmatrix}$$

$$c = \begin{pmatrix} 45 \\ 0 \\ 15 \\ 0 \\ 0 \end{pmatrix}$$

b)

$$A = \begin{pmatrix} -2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$b = \begin{pmatrix} 0 \\ 3 \\ 3 \\ 3 \end{pmatrix}$$

$$c = \begin{pmatrix} 15 \\ 41 \\ -11 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

2)

a) The optimal solution is either P1 or P6. It depends if it’s a minimization or maximization problem and if the objective function value at P1 is greater than the objective value at P6. In any case, one of these two vertexes will be the optimal solution.

b) The correct answer is “no”, because P1 and P7 are not adjacent feasible solutions. To solve the LP problem using Simplex one of the rules is improving the basic feasible solution by finding an adjacent feasible solution. Graphically, every vertex (the intersection of two lines) of the feasible region of this problem represents a basic solution which also implies only two constraints are active (which means the inequality constraints hold with equality). At each step of Simplex, we replace one basic variable by the new one, as a result, only one previous active constraint becomes inactive and only one previous inactive constraint becomes active. That’s why we can move only to adjacent basic feasible solutions. Note that graphically, all basic feasible solutions are vertexes of the feasible region (you can verify this claim by investigating a specified example if you feel it is hard to prove). And Simplex only explores the vertexes, i.e., basic feasible solutions. So P1’s adjacent basic feasible solutions are P4 and P8. We will never reach P2 and P3 in Simplex since they are not basic solutions. Also we cannot discuss P3’s adjacent basic feasible solutions in the context of Simplex since P3 is not a basic solution.

3)

Variables:

x_i : # of barrels of crude oil i processed in Fuel chain (i=1,2,3,4)
x_5 : # of barrels of crude oil 4 processed in Lube chain

(a) Constraints:

$$
\begin{aligned}
x_1 &\le 100{,}000\\
x_2 &\le 100{,}000\\
x_3 &\le 100{,}000\\
x_4 + x_5 &\le 200{,}000
\end{aligned}
$$

$$
\begin{aligned}
0.6x_1 + 0.5x_2 + 0.3x_3 + 0.4x_4 + 0.4x_5 &\le 170{,}000\\
0.2x_1 + 0.2x_2 + 0.3x_3 + 0.3x_4 + 0.1x_5 &\le 85{,}000\\
0.1x_1 + 0.2x_2 + 0.3x_3 + 0.2x_4 + 0.2x_5 &\le 85{,}000\\
0.2x_5 &\le 20{,}000
\end{aligned}
$$

$x_i \ge 0$, for all i = 1, ..., 5

Objective:

$$
\begin{aligned}
\text{Max } Z = {} & 45(0.6x_1 + 0.5x_2 + 0.3x_3 + 0.4x_4 + 0.4x_5)\\
& +30(0.2x_1 + 0.2x_2 + 0.3x_3 + 0.3x_4 + 0.1x_5)\\
& +15(0.1x_1 + 0.2x_2 + 0.3x_3 + 0.2x_4 + 0.2x_5)\\
& +60(0.2x_5) - (15+5)x_1 - (15+8)x_2 - (15+7.5)x_3\\
& -(25+3)x_4 - (25+2.5)x_5\\
= {} & 14.5x_1 + 8.5x_2 + 4.5x_3 + 2x_4 + 8.5x_5
\end{aligned}
$$

(b) Objective: Min $Z = 20x_1 + 23.5x_2 + 22.5x_3 + 28x_4 + 27.5x_5$

Constraints:

$$
\begin{aligned}
x_1 &\le 100{,}000\\
x_2 &\le 100{,}000\\
x_3 &\le 100{,}000\\
x_4 + x_5 &\le 200{,}000
\end{aligned}
$$

$$
\begin{aligned}
0.6x_1 + 0.5x_2 + 0.3x_3 + 0.4x_4 + 0.4x_5 &\ge 170{,}000\\
0.2x_1 + 0.2x_2 + 0.3x_3 + 0.3x_4 + 0.1x_5 &\ge 85{,}000\\
0.1x_1 + 0.2x_2 + 0.3x_3 + 0.2x_4 + 0.2x_5 &\ge 85{,}000\\
0.2x_5 &\ge 20{,}000
\end{aligned}
$$

$x_i \ge 0$, for all i = 1, ..., 5

4)

Variables:

x_1 - No. of nurses reporting to work in period 1
x_2 - No. of nurses reporting to work in period 2
x_3 - No. of nurses reporting to work in period 3
x_4 - No. of nurses reporting to work in period 4
x_5 - No. of nurses reporting to work in period 5
x_6 - No. of nurses reporting to work in period 6

Constraints:

$x_6 + x_1 \geq 60$
$x_1 + x_2 \geq 70$
$x_2 + x_3 \geq 60$
$x_3 + x_4 \geq 50$
$x_4 + x_5 \geq 20$
$x_5 + x_6 \geq 30$
$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Objective function:

Min $Z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$

5)

Let i=1, 2, 3 <=> months Jan, Feb and March respectively.

S_i = Bushels sold in month i
B_i = Bushels bought in month i
I_i = Bushels not sold in month i
M_i = Bushels not used in month i

Constraints:

Jan:
$$S_1 + I_1 = 1000 \quad \text{(selling)}$$
$$2.85B_1 + M_1 = 20000 \quad \text{(Cash balance)}$$
$$B_1 + I_1 \le 5000 \quad \text{(Storage)}$$

Feb:
$$S_2 + I_2 = B_1 + I_1$$
$$3.05B_2 + M_2 = M_1 + 3.1S_1$$
$$B_2 + I_2 \le 5000$$

Mar:
$$S3 + I_3 = B_2 + I_2$$
$$2.9\, B_3 + M_3 = M_2 + 3.25S_2$$
$$B_3 + I_3 = 2000$$

Objective function:

$$\text{Max } Z = M_3 + 2.95\, S_3$$

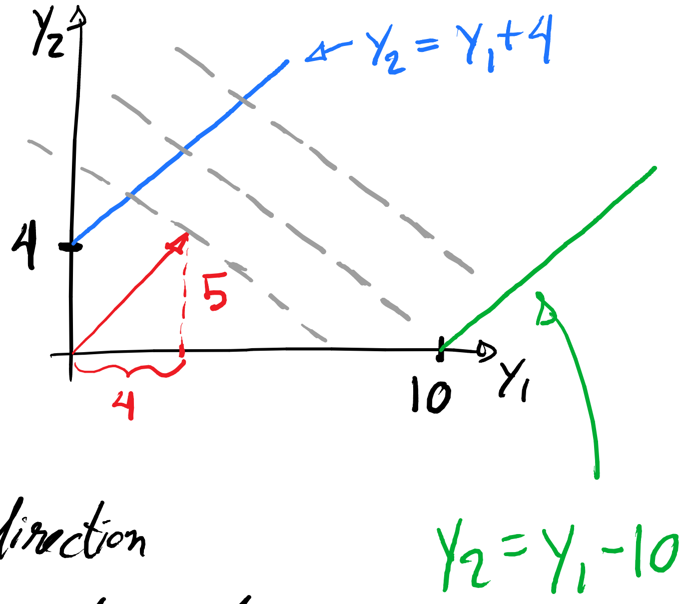
6:(a) max $4y_1 + 5y_2$

s.t.

$-y_1 + y_2 \leq 4$

$y_1 - y_2 \leq 10$

$y_1, y_2 \geq 0$



Note that moving in the direction $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ cannot cause any constraint violation. Also, moving in that direction increases our objective function value.

Consequently, the problem is unbounded.

(b) max $4y_1 + 5y_2$

$$\left.\begin{aligned} -y_1 + y_2 + y_3 \quad &= 4 \\ y_1 - y_2 \quad + y_4 &= 10 \end{aligned}\right\} \text{already canonical for basis } y_3, y_4$$

$y_1, y_2, y_3, y_4 \geq 0$

(c) $y_1 = y_2 = 0$, $y_3 = 4$, $y_4 = 10$

Check for optimality:

$$\tilde{c}_1 = 4 - [0 \;\; 0]\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 4 > 0 \quad \tilde{c}_2 = 5 - [0 \;\; 0]\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 5 > 0 \Rightarrow \text{solution is not optimal}$$

$\tilde{c}_2 = 5 > 4 = \tilde{c}_1 \Rightarrow y_2$ enters the basis

We use the minimum ratio rule to determine which which variable to remove from the basis.

Minimum Ratio Rule: Examine each constraint to determine how far the new basic variable may be increased. For those constraints in which the nonbasic variable has a positive coefficient, the limit is given by the ratio of the RHS constant to that positive coeficient.

For other constraints, the limit is set to ∞.

If a new basic variable has ∞ as its limit for all constraints, the problem is unbounded.

$$\Rightarrow y_2 = \min\{4/1, \infty\} = 4$$

Now we perform pivot operations to obtain the canonic form for the new basis

$$\left.\begin{aligned} -y_1 + y_2 + y_3 \quad &= 4 \\ y_1 - y_2 \quad + y_4 &= 10 \end{aligned}\right) + \quad \Rightarrow \quad y_3 + y_4 = 10$$

$$\boxed{\begin{aligned} -y_1 + y_2 + y_3 \quad &= 4 \\ y_3 + y_4 &= 10 \end{aligned}} \leftarrow \text{canonic form for } y_2, y_4$$

$$\tilde{c}_1 = 4 - [5 \ 0]\begin{bmatrix} -1 \\ 0 \end{bmatrix} = 9, \quad \tilde{c}_3 = 0 - [5 \ 0]\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -5$$

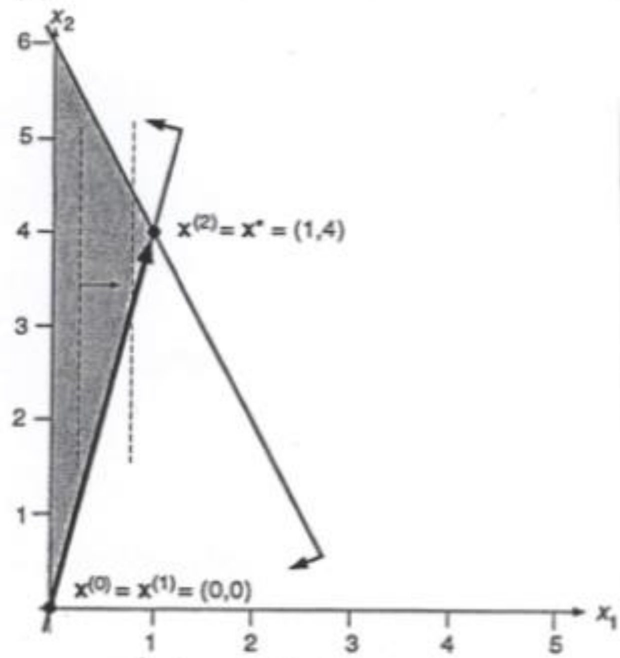
$\Rightarrow y_1$ enters the basis

$$y_1 = \min\{\infty, \infty\}$$

$\Rightarrow$ problem is unbounded

7)

(a)



(b) $\mathbf{A} = \begin{pmatrix} 6 & 3 & 1 & 0 \\ 12 & -3 & 0 & 1 \end{pmatrix}$, $\mathbf{b} = (18, 0)$, $\mathbf{c} = (1, 0, 0, 0)$

	x_1	x_2	x_3	x_4	
max $\mathbf{c}$	1	0	0	0	$\mathbf{b}$
$\mathbf{A}$	6	3	1	0	18
	12	-3	0	1	0
$t=0$	N	N	B	B	
$\mathbf{x}^{(0)}$	0	0	18	0	0
$\Delta\mathbf{x}, x_1$	1	0	-6	-12	$\bar{c}_1 = \boxed{1}$
$\Delta\mathbf{x}, x_2$	0	1	-3	3	$\bar{c}_2 = 0$
	-	-	$\frac{18}{6}$	$\boxed{\frac{0}{12}}$	$\lambda = 0$
$t=1$	B	N	B	N	
$\mathbf{x}^{(1)}$	0	0	18	0	0
$\Delta\mathbf{x}, x_2$	0.25	1	-4.50	0	$\bar{c}_2 = \boxed{0.25}$
$\Delta\mathbf{x}, x_4$	-0.08	0	0.50	1	$\bar{c}_4 = -0.08$
	-	-	$\boxed{\frac{18}{4.50}}$	-	$\lambda = 4$
$t=2$	B	B	N	N	
$\mathbf{x}^{(2)}$	1	4	0	0	1
$\Delta\mathbf{x}, x_3$	-0.06	-0.22	1	0	$\bar{c}_3 = -0.06$
$\Delta\mathbf{x}, x_4$	-0.06	0.11	0	1	$\bar{c}_4 = -0.06$

8)

a)

$$\max Z = 0x_1 + 45x_2 + 100x_3$$
$$x_1 + x_2 + x_3 \leq 10{,}000$$
$$x_2 - 0.5x_3 \geq 0$$
$$x_1 \geq 500$$

x_1: media tickets, x_2: university tickets, x_3: public tickets

The objective function maximizes ticket income. The first constraint limits sales to 10,000 tickets, the second assures at least half as many go to universities as to the general public and the third sets aside at least 500 tickets for the media.

b) Constraint 1: coefficients are number of seats per tickets
Constraint 2: coefficient of x3/coefficient of x2 is the negative of the minimum ratio between university tickets and general public tickets. Coefficient of x1 is zero, indicating this constraint does not involve x1
Constraint 3: coefficients of x2 and x3 are 0, indicating this constraint does not involve x2 and x3.

9)

(a) The marginal cost is the optimal dual variable value on the media constraint $v_3^* = 81.667$. **(b)** Both values are with the range $[500, \infty)$. Thus the additional revenue would be the extensions of the optimal dual rate or $(15000 - 10000)81.667 = \$408{,}335$, and $(20000 - 10000)81.667 = \$816{,}670$. **(c)** A reduction to \$50 is within the range $[45, \infty)$. Thus the revenue loss would be the extension of the primal rate or $(100 - 50)6333.333 = \$316{,}667$. A reduction to \$30 is outside the range. With objective function worsening hurting less and less, the loss would be at least the optimal primal rate extended to the end of the range or $(100 - 45)6333.333 = \$348{,}333$, and at most the extension to the new value or $(100 - 30)6333.333 = \$443{,}333$. **(d)** The new constraints would have the form $x_1 \leq .20x_2$ and $x_1 \leq .10x_2$, respectively. The first is satisfied by the current primal solution, because $(500) \leq .20(3166.667)$, so it would have no effect. The second is violated, because $(500) \not\leq .10(3166.667)$, so it would change the solution. **(e)** The new column would enter if its implicit cost with respect to the optimal dual solution is less than its revenue. With $.80(81.667) + 1(-36.667) = \$28.666$, the option would enter at \$35 per ticket, but not at \$25.

10)

a/b) X1: undergrad hours used, X2 : graduate hours used, X3: professional hours used. The objective function minimizes total cost. The first constraint assures at least 1,000 professional equivalent hours will be purchased. The second constraint enforces the limit on Proof's supervision time and the last restricts graduate hours to 500.

c)

input 1: 0.2 hours Proof supervision, $4 cost; output 1: 0.2 professional-equivalent hours programming;

input 2: 0.15 hours Proof supervision, 1 hour graduate maximum, $10 cost; output 2: 0.3 professional equivalent hours programming;

input 3: 0.15 hours Proof supervision, $25 cost; output 3: 1 professional equivalent hour programming.

11)

$$x_i = \begin{cases} 1, \textbf{\textit{ if a warehouse is located at site }} i = 1,2,3,4 \\ 0, \textbf{\textit{ otherwise}} \end{cases}$$

$$\min Z = \sum_{i=1}^{4} K_i x_i$$

$$s.t.$$

$$\textbf{\textit{For stores }} \boldsymbol{R_1} \textbf{\textit{ \& }} \boldsymbol{R_3}: \quad x_1 + x_2 \geq 1$$

$$\textbf{\textit{For stores }} \boldsymbol{R_2} \textbf{\textit{ \& }} \boldsymbol{R_4}: \quad x_1 + x_3 \geq 1$$

$$\textbf{\textit{For stores }} \boldsymbol{R_6} \textbf{\textit{ \& }} \boldsymbol{R_8}: \quad x_3 + x_4 \geq 1$$

$$\textbf{\textit{For stores }} \boldsymbol{R_7} \textbf{\textit{ \& }} \boldsymbol{R_9}: \quad x_2 + x_4 \geq 1$$

$$\textbf{\textit{For store }} \boldsymbol{R_5}: \quad x_1 + x_2 + x_3 + x_4 \geq 1$$

12)

$$\max W = 30y_1 + 20y_2$$

$$s.t.$$

$$y_1 + 2y_2 \leq 1$$

$$2y_1 + y_2 \leq 2$$

$$2y_1 + 3y_2 \leq 3$$

$$3y_1 + 2y_2 \leq 4$$

$$y_1, y_2 \geq 0$$

13) By inspection, x1 = 4, x2 = x3 = 0 is a feasible solution to the primal problem. For the dual problem:

$$\begin{aligned}
&Dual: \max W = 4y_1 + 3y_2 \\
&s.t. \\
&y_1 + y_2 \le 1 \\
&-y_2 \le -1 \\
&-y_1 + 2y_2 \le 1 \\
&y_1, y_2 \ge 0
\end{aligned}$$

The dual constraints are inconsistent since adding them all will produce

$$2y_2 \le 1$$

But the second constraint implies

$$y_2 \ge 1$$

Hence, by Corollary 5 of the Weak Duality Theorem, the Primal problem is unbounded.

14)

a)

$$\begin{aligned}
&Dual: \min W = 2y_1 + y_2 + 2y_3 \\
&s.t. \\
&y_1 + y_2 + 2y_3 \ge 1 \\
&y_1 - y_2 + y_3 \le 2 \\
&-y_1 + y_2 + y_3 = 1 \\
&y_1 \ge 0, y_2 \textbf{ \textit{is unrestricted in sign}}, y_3 \le 0
\end{aligned}$$

b)

By the Weak Duality Theorem,

$\max Z \le$ ***Value of W corresponding to same***

feasible solutions to Dual.

By inspection: $y_1 = 0, y_2 = 1, y_3 = 0$ ***is feasible for dual with*** $W = 1$.

Hence $\max Z \le 1$.

15)

$$\begin{aligned} \text{Max } z &= -4x_1 - 3x_2 \\ \text{sub to } \quad & x_1 + x_2 + x_3 = 1 \\ & -x_2 + x_4 = -1 \\ & -x1 + 2x_2 + x_5 = 1 \\ & x_1, \ldots, x_5 \geq 0 \end{aligned}$$

The basis (x_3, x_4, x_4) is dual feasible since all $\bar{c}_j \leq 0$. Applying the dual simplex metho we get the following :

	c_j	-4	-3	0	0	0	
C_B	Basis	x_1	x_2	x_3	x_4	x_5	b
0	x_3	1	1	1	0	0	1
0	x_4	0	-1	0	1	0	-1 <--
0	x_5	-1	2	0	0	1	1
$\bar{c}$	Row	-4	-3	0	0	0	Z=0
0	x_3	1	0	1	1	0	0
-3	x_2	0	1	0	-1	0	1
0	x_5	-1	0	0	2	1	-1 <--
$\bar{c}$	Row	-4	0	0	-3	0	Z=-3
0	x_3	0	0	1	3	1	-1 <--
-3	x_2	0	1	0	-1	0	0
-4	x_1	1	0	0	-2	-1	1
$\bar{c}$	Row	0	0	0	-11	-4	Z=-4

The min ratio rule fails at this stage since there are no negative elements in Row 1 to pivot. Hence, the given problem is infeasible.

16) a)

i)

$$\begin{aligned}
&\boldsymbol{min}\, 20 y_1 + 164 y_3 \\
&\boldsymbol{s.t.} \\
&y_1 + y_2 + 9 y_3 \geq 44 \\
&y_1 - y_2 - 3 y_3 \geq -3 \\
&y_1 + 0 y_2 + y_3 \geq 15 \\
&y_1 - 0 y_2 - y_3 \geq 56 \\
&y_1 : \textbf{\textit{unrestricted}}, y_2 \geq 0, y_3 \geq 0
\end{aligned}$$

ii)

Complementary Slackness conditions are trivially satisfied for the equality constraints, so we write them below only for the inequality constraints.

Primal:

$$y_2^* (-x_1^* + x_2^*) = 0$$
$$y_3^* \left(164 - 9x_1^* + 3x_2^* - x_3^* + x_4^*\right) = 0$$

Dual:

$$x_1^* \left(y_1^* + y_2^* + 9 y_3^* - 44\right) = 0$$
$$x_2^* \left(y_1^* - y_2^* - 3 y_3^* + 3\right) = 0$$
$$x_3^* \left(y_1^* + y_3^* - 15\right) = 0$$
$$x_4^* \left(y_1^* - y_3^* - 56\right) = 0$$

b)

i)

$$\max 19y_1 - 55y_2 + 7y_3$$
$$s.t.$$
$$y_1 + y_3 = 5$$
$$y_1 - 4y_2 + 6y_3 \le 1$$
$$y_1 - y_3 \le -4$$
$$y_1 - 8y_2 \ge 0$$
$$y_1 : \textit{unrestricted}, y_2 \ge 0, y_3 \ge 0$$

ii)

Primal:

$$y_2^*\left(55 - 4x_2^* - 8x_4^*\right) = 0$$
$$y_3^*\left(7 - x_1^* - 6x_2^* + x_3^*\right) = 0$$

Dual:

$$x_2^*\left(1 - y_1^* + 4y_2^* - 6y_3^*\right) = 0$$
$$x_3^*\left(-4 - y_1^* + y_3^*\right) = 0$$
$$x_4^*\left(y_1^* - 8y_2^*\right) = 0$$

c)

i)

$$\begin{aligned}
&\boldsymbol{min}\,15\boldsymbol{x}_1-4\boldsymbol{x}_3\\
&\boldsymbol{s.t.}\\
&11\boldsymbol{x}_1-\boldsymbol{x}_3\geq 19\\
&\boldsymbol{x}_1+\boldsymbol{x}_3\geq 4\\
&\boldsymbol{x}_1+\boldsymbol{x}_2=0\\
&5\boldsymbol{x}_2-\boldsymbol{x}_3=-8\\
&\boldsymbol{x}_1 : \boldsymbol{unrestricted}\,, \boldsymbol{x}_2\geq 0\,, \boldsymbol{x}_3\geq 0
\end{aligned}$$

ii)

Primal:

$$\begin{aligned}
&\boldsymbol{x}_2^*\left(-\boldsymbol{z}_1^*-5\boldsymbol{z}_2^*\right)=0\\
&\boldsymbol{x}_3^*\left(-4+\boldsymbol{y}_1^*-\boldsymbol{y}_2^*+\boldsymbol{z}_2^*\right)=0
\end{aligned}$$

Dual:

$$\begin{aligned}
&\boldsymbol{y}_1^*\left(11\boldsymbol{x}_1^*-\boldsymbol{x}_3^*-19\right)=0\\
&\boldsymbol{y}_2^*\left(\boldsymbol{x}_1^*+\boldsymbol{x}_3^*-4\right)=0
\end{aligned}$$

d)

i)

$$\begin{aligned}
&\boldsymbol{min}\,80\,y_1\\
&\boldsymbol{s.t.}\\
&y_1-y_2\geq 0\\
&y_1+2y_2-y_3\geq 0\\
&y_1+2y_3-y_4=10\\
&y_1+2y_4=10\\
&y_1 : \boldsymbol{unrestricted}\,, y_2\geq 0\,, y_3\geq 0\,, y_4\geq 0
\end{aligned}$$

ii)

Primal:

$$\begin{aligned}
&y_2^*\left(x_1^*-2x_2^*\right)=0\\
&y_3^*\left(x_2^*-2x_3^*\right)=0\\
&y_4^*\left(x_3^*-2x_4^*\right)=0
\end{aligned}$$

Dual:

$$\begin{aligned}
&x_1^*\left(y_1^*-y_2^*\right)=0\\
&x_2^*\left(y_1^*+2y_2^*-y_3^*\right)=0
\end{aligned}$$