Many problems in probability theory may be solved by simply counting the number of ways a certain event may occur.

We review some basic aspects of combinatorial analysis:

- combinations
- permutations
BASIC PRINCIPLE OF COUNTING

- Suppose that two experiments are to be performed:
  - experiment 1 may result in any one of the \( m \) possible outcomes
  - for each outcome of experiment 1, there exist \( n \) possible outcomes of experiment 2
- Therefore, there are \( mn \) possible outcomes of the two experiments

BASIC PRINCIPLE OF COUNTING

- The basic approach to prove this statement is via exhaustive enumeration: the set of outcomes for the two experiments is listed as:
  
  \[
  (1,1), (1,2), (1,3), \ldots (1,n) ; \\
  (2,1), (2,2), (2,3), \ldots (2,n) ; \\
  \vdots \\
  (m,1), (m,2), (m,3), \ldots (m,n)
  \]

  with \((i,j)\) denoting outcome \( i \) in experiment 1 and outcome \( j \) in experiment 2
**EXAMPLE 1: PAIR FORMATION**

- We need to form pairs of 1 boy and 1 girl by choosing from a group of 7 boys and 9 girls.
- There exist a total of \((7)(9) = 63\) possible pairs since there are 7 ways to pick a boy and 9 ways to pick a girl.

**GENERALIZED VERSION OF THE BASIC PRINCIPLE**

- For \(r\) experiments with the first experiment having \(n_1\) possible outcomes; for every outcome of the first experiment, there are \(n_2\) possible outcomes for the second experiment, and so on.
GENERALIZED VERSION OF THE BASIC PRINCIPLE

There are

$$\prod_{i=1}^{r} n_i = n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_r$$

possible outcomes for all the $r$ experiments, i.e.,

there are $\prod_{i=1}^{r} n_i$ possible branches in the illustration – high dimensionality even for a moderately small number $r$ of experiments

EXAMPLE 2: SUBCOMMITTEE CHOICES

The executive committee of an *Engineering Open House* function consists of:

- 3 first year students
- 4 sophomores
- 5 juniors
- 2 seniors

We need to form a subcommittee of 4 with each year represented: $3 \cdot 4 \cdot 5 \cdot 2 = 120$ different sub-committees are possible
EXAMPLE 3: LICENSE PLATE

- We consider possible combinations for a six-place license plate with the first three places consisting of letters and the last three places of numbers.
- Number of combinations with repeats allowed is
  \((26) (26) (26) (10) (10) (10) = 17,576,000\)
- Combination number if no repetition allowed is
  \((26) (25) (24) (10) (9) (8) = 11,232,000\)

EXAMPLE 4: \(n\) POINTS

- Consider \(n\) points at which we evaluate the function

\[ f(i) \in \{0,1\} \text{, } i = 1,2, \ldots, n \]

- Therefore, there are \(2^n\) possible function values.
PERMUTATIONS

A set of 3 objects \{A, B, C\} may be arranged in 6 different ways:

- BCA
- ABC
- CBA
- BAC
- ACB
- CAB

Each arrangement is called a permutation.

The total number of permutations is derived from the basic principle:

- there are 3 ways to pick the first element
- there are 2 ways to pick the second element
- there is 1 way to pick the third element

Therefore, there are \(3 \times 2 \times 1 = 6\) ways to arrange the 3 elements.

In general, a set of \(n\) objects can be arranged into different permutations:

\[n! = n(n-1)(n-2) \ldots 1\]
EXAMPLE 5: BASEBALL

- Number of possible batting orders for a baseball team with nine members is
  \[ 9! = 362,880 \]
- Suppose that the team, however, has altogether 12 members; the number of possible batting orders is the product of the number of team formations and the number of permutations is
  \[ \frac{12!}{3! \cdot 9!} = \frac{12!}{3!} = 2 \cdot (11!) = 79,833,600 \]

EXAMPLE 6: CLASSROOM

- A class with 6 boys and 4 girls is ranked in terms of weight; assume that no two students have the same weight
- There are
  \[ 10! = 3,628,800 \]
  possible rankings
- If the boys (girls) are ranked among themselves, the number of different possible rankings is
  \[ 6!4! = 17,280 \]
EXAMPLE 7: BOOKS

- A student has 10 books to put on the shelf:
  - 4 EE, 3 Math, 2 Econ, and 1 Decision
- Student arranges books so that all books in each category are grouped together
- There are $4!3!2!1!$ arrangements so that all EE books are first in line, then the Math books, Econ books, and Decision book

EXAMPLE 8: BOOKS

- But, there are $4!$ possible orderings of the subjects
- Therefore, there are

$$4!4!3!2!1! = 6,912$$

permutations of arranging the 10 books
EXAMPLE 9: PEPPER

We wish to determine the number of different letter arrangements in the word *PEPPER*.

Consider first the letters $P_1 E_1 P_2 P_3 E_2 R$ where we distinguish the repeated letters among themselves: there are $6!$ permutations of the 6 distinct letters.

However, if we consider any single permutation of the 6 letters – for example, $P_1 P_2 E_1 P_3 E_2 R$ – provides the same word *PEPPER* as 11 other permutations if we fail to distinguish between the same letters. Therefore, there are $6!$ permutations for distinct letters but only

$$\frac{6!}{3!2!} = 60$$

permutations with repeated letters that are not distinct, *i.e.*, without subscripts on repeated letters.
GENERAL STATEMENT

- Consider a set of $n$ objects in which
  - $n_1$ are alike (category 1)
  - $n_2$ are alike (category 2)
  - $\ldots$
  - $n_r$ are alike (category $r$)

- There are
  $$\frac{n!}{n_1!n_2!\ldots n_r!}$$
  different permutations

EXAMPLE 9: COLORED BALLS

- We have 3 white, 4 red, and 4 black balls which we arrange in a row; similarly colored balls are indistinguishable from each other

- There are
  $$\frac{11!}{3!4!4!} = 11,550$$
  possible arrangements
COMBINATIONS

- Given \( n \) objects, we form groups of \( r \) objects and determine the number of different groups we can form.

- For example, consider 5 objects denoted as \( A, B, C, D \) and \( E \) and form groups of 3 objects:
  - we can pick the first item in exactly 5 ways
  - we can pick the second item in exactly 4 ways
  - we can pick the third item in exactly 3 ways

and, therefore, we can select

\[
5 \cdot 4 \cdot 3 = 60
\]

possible groups in which the order of the groups is taken into account.

- But, if the order of the objects is not of interest, i.e., we ignore that each group of three objects can be arranged in 6 different permutations, the total number of distinct groups is

\[
\frac{5!}{2!3!} = \frac{60}{6} = 10
\]
GENERAL STATEMENT ON COMBINATIONS

- The objective is to arrange \( n \) elements into groups of \( r \) elements.
- We can select groups of \( r \) elements in \( \frac{n!}{(n-r)!} \) different ways.
- But, each group of \( r \) has \( r! \) permutations.
- The number of different combinations is \( \frac{n!}{(n-r)!r!} \).

BINOMIAL COEFFICIENTS

- We define the term \( \binom{n}{r} \) as the binomial coefficient of \( n \) and \( r \).
- A binomial coefficient gives the number of possible combinations of \( n \) elements taken \( r \) at a time.
EXAMPLE 10: COMMITTEE SELECTION

We wish to select three persons to represent a class of 40: how many groups of 3 can be formed?

There are possible group selections

\[
\frac{40!}{37!3!} = \frac{40 \cdot 39 \cdot 38}{3 \cdot 2 \cdot 1} = 20 \cdot 13 \cdot 38 = 9,880
\]

EXAMPLE 11: GROUP FORMATION

Given a group of 5 boys and 7 girls, form sets consisting of 2 boys and 3 girls

There are possible ways to form such groups

\[
\binom{5}{2} \binom{7}{3} = \frac{5!}{3!2!} \frac{7!}{4!3!} = \frac{5 \cdot 4}{2} \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} = 350
\]
**GENERAL COMBINATORIAL IDENTITY**

\[
\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}
\]

- Number of ways of selecting groups of \(r\) from \(n\)
- Number of ways of selecting groups of \(r-1\) from \(n-1\)
- Number of ways of selecting groups of \(r\) from \(n-1\)

**MULTINOMIAL COEFFICIENTS**

- Given a set of \(n\) distinct items, form \(r\) distinct groups of respective sizes \(n_1, n_2, \ldots, n_r\) with
  \[\sum_{i=1}^{r} n_i = n\]
- There are
  \[\binom{n}{n_1}\]
  possible choices for the first group
MULTINOMIAL COEFFICIENTS

For each choice of the first group, there are
\[
\binom{n - n_1}{n_2}
\]
possible choices for the second group

We continue with this reasoning and we conclude that there are
\[
\frac{n!}{n_1! n_2! \ldots n_r!}
\]
possible groups

MULTINOMIAL COEFFICIENTS

The previous conclusion was gained by realizing that
\[
\binom{n}{n_1} \binom{n - n_1}{n_2} \binom{n - n_1 - n_2}{n_3} \cdots \binom{n - n_1 - n_2 - \ldots - n_{r-1}}{n_r} = \frac{n!}{(n - n_1)! n_1!} \frac{(n - n_1)!}{(n - n_1 - n_2)! n_2!} \cdots \frac{n - n_1 - n_2 - \ldots - n_{r-1}}{\theta! n_r!} = \frac{n!}{n_1! n_2! \ldots n_r!}
\]
MULTINOMIAL COEFFICIENTS

Let

\[ n = n_1 + n_2 + n_3 + \ldots + n_r \]

we define the \textit{multinomial coefficient}

\[ \binom{n}{n_1, n_2, \ldots, n_r} \triangleq \frac{n!}{n_1!n_2!n_3! \ldots n_r!} \]

A multinomial coefficient represents the number of possible allocations of \( n \) distinct objects into \( r \) distinct groups of respective sizes \( n_1, n_2, \ldots, n_r \).

EXAMPLE 12: POLICE

A police department of a small town has 10 officers.

The department policy is to have 5 members on street patrol, 2 members at the station and 3 on reserve.

The number of possible allocations is

\[ \frac{10!}{5!3!2!} = 2,520 \]
EXAMPLE 13: TEAM FORMATION

- We need to form two teams, the *A* team and the *B* team, with each team having 5 *boys* from a group of 10 *boys*.
- There are
  \[
  \frac{10!}{5!5!} = 252
  \]
  possible teams.

EXAMPLE 13: TEAM FORMATION

- Suppose that these two teams are to play against one another.
- In this case, the order of the two teams is irrelevant since there is no team *A* and team *B* per se but simply a division of 10 *boys* into 2 groups of 5 each.
- The number of ways to form the two teams is
  \[
  \frac{1}{2!} \left( \frac{10!}{5!5!} \right) = 126
  \]
EXAMPLE 14: TEA PARTY

- A woman has 8 friends of whom she will invite 5 to a tea party.

- How many choices does she have if 2 of the friends are feuding and refuse to attend together?

- How many choices does she have if 2 of her friends will only attend together?