ECE 307 – Techniques for Engineering Decisions

Lecture 9. Review of Combinatorial Analysis

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COMBINATORIAL ANALYSIS

- Many problems in probability theory may be solved by simply counting the number of ways a certain event may occur
- We review some basic aspects of combinatorial analysis
  - combinations
  - permutations

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BASIC PRINCIPLE OF COUNTING

- Suppose that two experiments are to be performed:
  - experiment 1 may result in any one of the m possible outcomes
  - for each outcome of experiment 1, there exist n possible outcomes of experiment 2
- Therefore, there are mn possible outcomes of the two experiments

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BASIC PRINCIPLE OF COUNTING

- The *basic approach* to prove this statement is via exhaustive enumeration: the set of outcomes for the two experiments is listed as:

$$\begin{array}{l} (1,1)\ ,\ (1,2)\ ,\ (1,3),\ \ldots\ (1,n)\ ; \\ (2,1)\ ,\ (2,2)\ ,\ (2,3),\ \ldots\ (2,n)\ ; \\ \quad\vdots \qquad\qquad\qquad\qquad \vdots \\ (m,1),(m,2),(m,3),\ \ldots\ (m,n) \end{array}$$

with $(i\,,j)$ denoting outcome i in experiment 1 and outcome j in experiment 2

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EXAMPLE 1: PAIR FORMATION

- We need to form pairs of 1 boy and 1 girl by choosing from a group of 7 boys and 9 girls
- There exist a total of $(7)(9) = 63$ possible pairs since there are 7 ways to pick a boy and 9 ways to pick a girl

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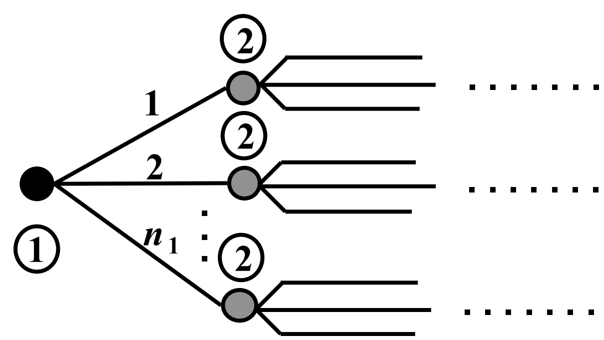
GENERALIZED VERSION OF THE *BASIC PRINCIPLE*

- For r experiments with the first experiment having n_1 possible outcomes; for every outcome of the first experiment, there are n_2 possible outcomes for the second experiment, and so on



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GENERALIZED VERSION OF THE BASIC PRINCIPLE

- There are

$$\prod_{i=1}^{r} n_i = n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_r$$

possible outcomes for all the r experiments, i.e., there are $\prod_{i=1}^{r} n_i$ possible branches in the illustration – high dimensionality even for a moderately small number r of experiments

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EXAMPLE 2: SUBCOMMITTEE CHOICES

- The executive committee of an *Engineering Open House* function consists of:
  - 3 first year students
  - 4 sophomores
  - 5 juniors
  - 2 seniors
- We need to form a subcommittee of 4 with each year represented: $3 \cdot 4 \cdot 5 \cdot 2 = 120$ different sub-committees are possible

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EXAMPLE 3: LICENSE PLATE

- We consider possible combinations for a six-place license plate with the first three places consisting of letters and the last three places of numbers
- Number of combinations with repeats allowed is

$$(26)\,(26)\,(26)\,(10)\,(10)\,(10) = 17{,}576{,}000$$

- Combination number if no repetition allowed is

$$(26)\,(25)\,(24)\,(10)\,(9)\,(8) = 11{,}232{,}000$$

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EXAMPLE 4: n POINTS

- Consider n points at which we evaluate the function

$$f(i) \in \{ 0,1 \}, \; i = 1, 2, \ldots, n$$

- Therefore, there are 2^n possible function values

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PERMUTATIONS

- A set of 3 objects $\{A, B, C\}$ may be arranged in 6 different ways:

BCA	*ABC*	*CBA*
BAC	*ACB*	*CAB*

- Each arrangement is called a *permutation*
- The total number of permutations is derived from the *basic principle*:
  - there are 3 ways to pick the first element
  - there are 2 ways to pick the second element
  - there is 1 way to pick the third element

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PERMUTATIONS

- Therefore, there are $3 \cdot 2 \cdot 1 = 6$ ways to arrange the 3 elements
- In general, a set of n objects can be arranged into

$$n! = n(n-1)(n-2) \ldots 1$$

different permutations

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EXAMPLE 5: BASEBALL

- ❑ Number of possible batting orders for a baseball team with nine members is

$$9! = 362,880$$

- ❑ Suppose that the team, however, has altogether 12 members; the number of possible batting orders is the product of the number of team formations and the number of permutations is

$$\frac{12!}{3!\,9!} \bullet 9! = \frac{12!}{3!} = 2(11!) = 79,833,600$$

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EXAMPLE 6: CLASSROOM

- ❑ A class with 6 boys and 4 girls is ranked in terms of weight; assume that no two students have the same weight
- ❑ There are

$$10! = 3,628,800$$

possible rankings

- ❑ If the boys (girls) are ranked among themselves, the number of different possible rankings is

$$6!4! = 17,280$$

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EXAMPLE 7: BOOKS

- A student has 10 books to put on the shelf:

  4 *EE*, 3 *Math*, 2 *Econ*, and 1 *Decision*
- Student arranges books so that all books in each category are grouped together
- There are $4!3!2!1!$ arrangements so that all *EE* books are first in line, then the *Math* books, *Econ* books, and *Decision* book

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EXAMPLE 8: BOOKS

- But, there are $4!$ possible orderings of the subjects
- Therefore, there are

$$4!4!3!2!1! = 6{,}912$$

permutations of arranging the 10 books

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EXAMPLE 9: PEPPER

- We wish to determine the number of different letter arrangements in the word *PEPPER*
- Consider first the letters $P_1 E_1 P_2 P_3 E_2 R$ where we distinguish the repeated letters among themselves: there are $6!$ permutations of the 6 distinct letters

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EXAMPLE 9: PEPPER

- However, if we consider any single permutation of the 6 letters – for example, $P_1 P_2 E_1 P_3 E_2 R$ – provides the same word *PPEPER* as 11 other permutations if we fail to distinguish between the same letters
- Therefore, there are $6!$ permutations for distinct letters but only

$$\frac{6!}{3!2!} = 60$$

permutations with repeated letters that are not distinct, *i.e.*, without subscripts on repeated letters

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GENERAL STATEMENT

- Consider a set of n objects in which

  n_1 are alike (category 1)

  n_2 are alike (category 2)

  ⋮

  n_r are alike (category r)

- There are

$$\frac{n!}{n_1!n_2!.....n_r!}$$

  different permutations

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EXAMPLE 9: COLORED BALLS

- We have 3 *white*, 4 *red*, and 4 *black* balls which we arrange in a row; similarly colored balls are indistinguishable from each other

- There are

$$\frac{11!}{3!4!4!} = 11{,}550$$

  possible arrangements

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COMBINATIONS

- Given n objects, we form groups of r objects and determine the number of different groups we can form
- For example, consider 5 objects denoted as A,B,C,D and E and form groups of 3 objects:
  - we can pick the first item in exactly 5 ways
  - we can pick the second item in exactly 4 ways
  - we can pick the third item in exactly 3 ways

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COMBINATIONS

and, therefore, we can select

$$5 \cdot 4 \cdot 3 = 60$$

possible groups in which the order of the groups is taken into account

- But, if the order of the objects is not of interest, i.e., we ignore that each group of three objects can be arranged in 6 different permutations, the total number of distinct groups is

$$\frac{5!}{2!3!} = \frac{60}{6} = 10$$

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GENERAL STATEMENT ON COMBINATIONS

- The objective is to arrange n elements into groups of r elements
- We can select groups of r elements

$$\frac{n!}{(n-r)!}$$

  different ways
- But, each group of r has $r\,!$ permutations
- The number of different combinations is

$$\frac{n!}{(n-r)!r!}$$

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BINOMIAL COEFFICIENTS

- We define the term

$$\binom{n}{r} \triangleq \frac{n!}{(n-r)!r!}$$

  as the *binomial coefficient* of n and r

- A binomial coefficient gives the number of possi– ble combinations of n elements taken r at a time

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EXAMPLE 10: COMMITTEE SELECTION

- We wish to select three persons to represent a class of 40: how many groups of 3 can be formed?
- There are

$$\frac{40!}{37!3!} = \frac{40 \cdot 39 \cdot 38}{3 \cdot 2 \cdot 1} = 20 \cdot 13 \cdot 38 = 9{,}880$$

possible group selections

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EXAMPLE 11: GROUP FORMATION

- Given a group of 5 *boys* and 7 *girls*, form sets consisting of 2 *boys* and 3 *girls*
- There are

$$\binom{5}{2}\binom{7}{3} = \frac{5!}{3!\,2!}\frac{7!}{4!\,3!} = \frac{5 \cdot 4}{2}\;\frac{7 \cdot 6 \cdot 5}{3 \cdot 2} = 350$$

possible ways to form such groups

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GENERAL COMBINATORIAL IDENTITY

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

number of ways of selecting groups of r from n

number of ways of selecting groups of $r-1$ from $n-1$

number of ways of selecting groups of r from $n-1$

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MULTINOMIAL COEFFICIENTS

- Given a set of n distinct items, form r distinct groups of respective sizes $n_1, n_2, \ldots,$ and n_r with

$$\sum_{i=1}^{r} n_i = n$$

- There are

$$\binom{n}{n_1}$$

possible choices for the first group

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MULTINOMIAL COEFFICIENTS

- For each choice of the first group, there are

$$\begin{pmatrix} n - n_1 \\ n_2 \end{pmatrix}$$

possible choices for the second group

- We continue with this reasoning and we conclude that there are

$$\frac{n!}{n_1!n_2! \ldots n_r!}$$

possible groups

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MULTINOMIAL COEFFICIENTS

- The previous conclusion was gained by realizing that

$$\begin{pmatrix} n \\ n_1 \end{pmatrix}\begin{pmatrix} n - n_1 \\ n_2 \end{pmatrix}\begin{pmatrix} n - n_1 - n_2 \\ n_3 \end{pmatrix} \cdots \begin{pmatrix} n - n_1 - n_2 - \cdots n_{r-1} \\ n_r \end{pmatrix} =$$

$$\frac{n!}{(n-n_1)!n_{1!}} \; \frac{(n-n_1)!}{(n-n_1-n_2)!n_2!} \; \cdots \; \frac{n - n_1 - n_2 - \ldots n_{r-1}}{0!n_r!} =$$

$$\frac{n!}{n_1!n_2! \ldots n_r!}$$

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MULTINOMIAL COEFFICIENTS

- Let

$$n = n_1 + n_2 + n_3 + \ldots + n_r$$

we define the *multinomial coefficient*

$$\binom{n}{n_1, n_2, \ldots, n_r} \triangleq \frac{n!}{n_1! n_2! n_3! \ldots n_r!}$$

- A multinomial coefficient represents the number of possible allocations of n distinct objects into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$

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EXAMPLE 12: POLICE

- A police department of a small town has 10 officers
- The department policy is to have 5 members on street patrol, 2 members at the station and 3 on reserve
- The number of possible allocations is

$$\frac{10!}{5!3!2!} = 2{,}520$$

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EXAMPLE 13: TEAM FORMATION

- We need to form two teams, the A team and the B team, with each team having 5 *boys* from a group of 10 *boys*
- There are

$$\frac{10!}{5!5!} = 252$$

possible teams

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EXAMPLE 13: TEAM FORMATION

- Suppose that these two teams are to play against one another
- In this case, the order of the two teams is irrelevant since there is no team A and team B per se but simply a division of 10 *boys* into 2 groups of 5 each
- The number of ways to form the two teams is

$$\frac{1}{2!}\left(\frac{10!}{5!5!}\right) = 126$$

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EXAMPLE 14: TEA PARTY

- A woman has 8 friends of whom she will invite 5 to a tea party
- How many choices does she have if 2 of the friends are feuding and refuse to attend together?
- How many choices does she have if 2 of her friends will only attend together?

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