## OPTIMAL CUTTING STOCK PROBLEM

- A paper company gets an order for:
  - 8 rolls of 2 ft paper sold at 2.50 $/roll
  - 6 rolls of 2.5 ft paper sold at 3.10 $/roll
  - 5 rolls of 4 ft paper sold at 5.25 $/roll
  - 4 rolls of 3 ft paper sold at 4.40 $/roll

- The company only has 13 ft of paper to fill these orders; partial orders may be filled with full rolls.

- Determine how to fill orders to maximize revenues.
**DP SOLUTION APPROACH**

- **Stage** is an order and since there are 4 orders we construct a 4-stage DP

```
stage 4  stage 3  stage 2  stage 1
  ↓     ↓     ↓     ↓
  R4    R3    R2    R1

s4 → d4 → s3 → d3 → s2 → d2 → s1 → d1 → s0
```

- A **state** in stage n is the remaining ft of paper left for the order being processed at stage n and all the remaining stages

- A decision in stage n is the amount of rolls to produce in stage n:
**DP SOLUTION APPROACH**

\[ d_n = \left\lfloor \frac{F_0}{L_n} \right\rfloor, \text{ the largest integer in } \frac{F_0}{L_n} \]

where,

\[ L_n = \text{length of order } n \text{ (ft)} \]

\[ F_0 = \text{length of available paper (ft)} \]

- The *return function* at stage \( n \) is the additional revenues gained from producing \( d_n \) rolls.

**DP SOLUTION APPROACH**

- The *transition function* measures amount of paper remaining at stage \( n \)

\[ s_{n-1} = s_n - d_n L_n \quad n = 2, 3, 4 \]

\[ s_0 = s_1 - d_1 L_1 \]

and \( s_0 \) needs to be as close as possible to \( \theta \)

- Therefore, we set

\[ d_1 = \left\lfloor \frac{s_1}{L_1} \right\rfloor \]
**DP SOLUTION APPROACH**

- The recursion relation is
  \[
  f_n^*(s_n) = \max \left\{ R_n(s_n, d_n) + f_{n-1}^*(s_{n-1}) \right\}
  \]
  \[
  0 \leq d_n \leq \left[ \frac{s_n}{L_n} \right]
  \]
  where
  \[
  s_{n-1} = s_n - d_n L_n
  \]
  and
  \[
  f_0^*(s_0) = 0
  \]
  \[
  f_n(s_n, d_n) = r_n d_n + f_{n-1}^*(s_n - d_n L_n), \quad n = 1, 2, 3, 4
  \]

**DP SOLUTION APPROACH**

- We arbitrarily order the stages and pick

<table>
<thead>
<tr>
<th>stage n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>length of order (ft)</td>
<td>2.5</td>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

- We proceed backwards from stage 1 to stage 4

and we know that
DP SOLUTION: STAGE 1

\[ f_1^*(s_1) = \max_{0 \leq d_1 \leq 5} \left\{ r_1(s_1, d_1) \right\} = \max_{0 \leq d_1 \leq 5} \left\{ 3.10 \cdot d_1 \right\} \]

\[ d_1 \leq \left[ \frac{13}{2.5} \right] = 5 \]

<table>
<thead>
<tr>
<th>( s_1 )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d_1 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( f_1^*(s_1) )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( d_1^* )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

DP SOLUTION: STAGE 2

\[ f_2^*(s_2) = \max_{0 \leq d_2 \leq 3} \left\{ 5.25 \cdot d_2 + f_1^*(s_2 - 4 \cdot d_2) \right\} \]

\[ d_2 \leq \left[ \frac{13}{4} \right] = 3 \]

<table>
<thead>
<tr>
<th>( s_2 )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d_2 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3.10</td>
<td>3.10</td>
<td>6.20</td>
<td>6.20</td>
<td>6.20</td>
<td>9.30</td>
<td>9.30</td>
<td>12.40</td>
<td>12.40</td>
<td>12.40</td>
<td>15.50</td>
</tr>
<tr>
<td>( f_2^*(s_2) )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3.10</td>
<td>5.25</td>
<td>6.20</td>
<td>6.20</td>
<td>8.35</td>
<td>11.45</td>
<td>14.55</td>
<td>14.55</td>
<td>14.55</td>
<td>14.55</td>
<td>14.55</td>
</tr>
<tr>
<td>( d_2^* )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>
**DP SOLUTION: STAGE 3**

\[ f^*(s_3) = \max_{0 \leq d_3 \leq 4} \left\{ 4.40 \ d_3 + f^*(s_3 - 3 \ d_3) \right\} \]

\[ d_3 \leq \left[ \frac{13}{3} \right] = 4 \]

<table>
<thead>
<tr>
<th>(d_3)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>4.40</td>
<td>4.40</td>
<td>4.40</td>
<td>7.50</td>
<td>9.65</td>
<td>10.60</td>
<td>10.60</td>
<td>12.75</td>
<td>14.90</td>
<td>15.85</td>
<td>16.80</td>
</tr>
<tr>
<td>1</td>
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<td>-</td>
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<td>8.80</td>
<td>8.80</td>
<td>8.80</td>
<td>11.90</td>
<td>14.05</td>
<td>15.00</td>
<td>15.00</td>
<td>17.15</td>
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<td>2</td>
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<td>13.20</td>
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</tbody>
</table>

The maximum profits are \$18.45

**DP SOLUTION: STAGE 4**

\[ f^*(s_4) = \max_{0 \leq d_4 \leq 6} \left\{ 2.5 \ d_4 + f^*(s_4 - 2 \ d_4) \right\} \]

\[ d_4 \leq \left[ \frac{13}{2} \right] = 6 \]

<table>
<thead>
<tr>
<th>(d_4)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>(d_4^*)</th>
<th>(f^*(s_4))</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>15.00</td>
<td>15.00</td>
<td>15.00</td>
<td>15.00</td>
<td>15.00</td>
<td>15.00</td>
<td>15.00</td>
<td>18.45</td>
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<tr>
<td>3</td>
<td>17.15</td>
<td>17.15</td>
<td>17.15</td>
<td>17.15</td>
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<td>17.15</td>
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<td>4</td>
<td>18.45</td>
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<td>6</td>
<td>18.45</td>
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<td>7</td>
<td>18.45</td>
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<td>8</td>
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<td>9</td>
<td>18.45</td>
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<td>10</td>
<td>18.45</td>
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<td>11</td>
<td>18.45</td>
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<td>18.45</td>
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<tr>
<td>12</td>
<td>18.45</td>
<td>18.45</td>
<td>18.45</td>
<td>18.45</td>
<td>18.45</td>
<td>18.45</td>
<td>18.45</td>
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<td></td>
</tr>
<tr>
<td>13</td>
<td>18.45</td>
<td>18.45</td>
<td>18.45</td>
<td>18.45</td>
<td>18.45</td>
<td>18.45</td>
<td>18.45</td>
<td>18.45</td>
<td></td>
</tr>
</tbody>
</table>

\(\square\) The maximum profits are \$18.45
DP OPTIMAL SOLUTION

- The *optimal* solution is obtained by retracing

\[
\begin{align*}
    f_1^*(s_1 = 0) &= 0 \quad \text{with} \quad d_1^* = 0 \quad \leftrightarrow \text{no rolls of 2.5 ft} \\
    f_2^*(s_2 = 4) &= 5.25 \quad \text{with} \quad d_2^* = 1 \quad \leftrightarrow \text{1 roll of 4 ft} \\
    f_3^*(s_3 = 13) &= 18.45 \quad \text{with} \quad d_3^* = 3 \quad \leftrightarrow \text{3 rolls of 3 ft} \\
    f_4^*(s_4 = 13) &= 18.45 \quad \text{with} \quad d_4^* = 0 \quad \leftrightarrow \text{no rolls of 2 ft}
\end{align*}
\]

SENSITIVITY CASE

- Consider the situation that, owing to an incorrect measurement, in truth, there are only 11 ft available for the rolls.

- We note that the solution for the original 13 ft covers this possibility in the *stages* 1, 2 and 3 but we need to re-compute the results of *stage 4*, which we now call *stage 4'*
SENSITIVITY CASE: \( \textit{STAGE} 4' \)

- The \( \textit{stage} \ 4' \) computations become
  \[
  d_{4'} \leq \left[ \frac{11}{2} \right] = 5
  \]

<table>
<thead>
<tr>
<th>( d_{4'} )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>( d_{4'}^* )</th>
<th>( f_{4'}^*(s_4) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_4 = 11 )</td>
<td>15</td>
<td>15.7</td>
<td>14.65</td>
<td>13.7</td>
<td>14.4</td>
<td>12.5</td>
<td>1</td>
<td>15.7</td>
</tr>
</tbody>
</table>

- The \( \textit{optimal} \) profits in this sensitivity case are $15.7

---

SENSITIVITY CASE \( \textit{OPTIMUM} \)

- The retrace of the solution path obtains
  \[
  d_{4'}^* = 1 \iff 1 \text{ roll of } 2 \text{ ft}
  \]
  \[
  d_{3'}^* = 3 \iff 3 \text{ rolls of } 3 \text{ ft}
  \]
  \[
  d_{2'}^* = 0 \iff 0 \text{ rolls of } 4 \text{ ft}
  \]
  \[
  d_{1'}^* = 0 \iff 0 \text{ rolls of } 2.5 \text{ ft}
  \]

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ANOTHER SENSITIVITY CASE

- We consider the case with the initial 13 ft, but in addition we get the constraint that at least 1 roll of 2 ft must be produced:

\[ d_4 \geq 1 \]

- We note that no additional work is needed since the computations in the first tables have all the required data

- This sensitivity case optimum profits are $18.2

- The optimal solution is:

\[ f^*_{4^*}(s_4 = 13) = 18.2 \quad \text{with} \quad d_{4^*}^* = 2 \leftrightarrow 2 \text{ rolls of 2 ft} \]

\[ f^*_{3^*}(s_3 = 9) = 13.2 \quad \text{with} \quad d_{3^*}^* = 3 \leftrightarrow 3 \text{ rolls of 3 ft} \]

and since \( s_2 = s_1 = 0 \quad d_{2^*}^* = 0 \leftrightarrow 0 \text{ rolls of 4 ft} \]

\[ d_{1^*}^* = 0 \leftrightarrow 0 \text{ rolls of 2.5 ft} \]

- The additional constraint reduces the optimum from $18.45 to $18.2 and so lowers revenues $0.25
INVENTORY CONTROL PROBLEM

- This problem is concerned with the development of an *optimal* ordering policy for a retailer.

- The sales of a seasonal item has the demands:

<table>
<thead>
<tr>
<th>month</th>
<th>Oct</th>
<th>Nov</th>
<th>Dec</th>
<th>Jan</th>
<th>Feb</th>
<th>Mar</th>
</tr>
</thead>
<tbody>
<tr>
<td>demand</td>
<td>40</td>
<td>20</td>
<td>30</td>
<td>40</td>
<td>30</td>
<td>20</td>
</tr>
</tbody>
</table>

INVENTORY CONTROL PROBLEM

- All units sold are purchased from a vendor at 4 $/unit; units are sold in lots of 10, 20, 30, 40 or 50 with the corresponding discount:

<table>
<thead>
<tr>
<th>lot size</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>discount %</td>
<td>4</td>
<td>5</td>
<td>10</td>
<td>20</td>
<td>25</td>
</tr>
</tbody>
</table>
INVENTORY CONTROL PROBLEM

- There are additional ordering costs: each order incurs fixed costs of $2 and $8 for shipping, handling and insurance.
- The storage limitations of the retailer require that no more than 40 units be in inventory at the end of the month and the storage charges are 0.2 $/unit; there is 0 inventory at the beginning and at the end of the period under consideration.
- Underlying assumption: demand occurs at a constant rate throughout each month.

DP SOLUTION APPROACH

- We formulate the problem as a DP and use a backward process to solve.
- Each stage corresponds to a month.

<table>
<thead>
<tr>
<th>month</th>
<th>Oct</th>
<th>Nov</th>
<th>Dec</th>
<th>Jan</th>
<th>Feb</th>
<th>Mar</th>
</tr>
</thead>
<tbody>
<tr>
<td>stage n</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>
$R_n$ is the contribution to the total cost of the ordering policy from the stage $n$ decision, $n = 1, 2, ..., 6$

The state variable $s_n$ in stage $n$ is defined as the amount of inventory that is stored from the previous month, taking into account that $n$ additional months remain in the planning period – the month corresponding to stage $n$ plus the months in the stages $n - 1$, $n - 2$, ..., 1
**DP SOLUTION APPROACH**

- The decision variable $d_n$ in stage $n$ is the amount of units ordered to satisfy the $n$ remaining months’ demands $D_n$ and $D_i$, $i = n-1, n-2, \ldots, 2, 1$.

- The transition function is defined by:

\[
\begin{align*}
  s_{n-1} &= s_n + d_n - D_n & n = 1, 2, \ldots, 6 \\
  s_0 &= 0 \\
  s_6 &= 0 \\
\end{align*}
\]

- Demand in month $n$.

---

**DP SOLUTION APPROACH**

- The return function in the stage $n$ is given by:

\[
r_n(s_n, d_n) = \phi(d_n) + h_n(s_n + d_n - D_n)
\]

- Ordering costs and storage costs.

with

\[
d_n = 0, 10, 20, 30, 40 \text{ or } 50
\]

\[
\phi(d_n) = 10 + 4[1 - \rho(d_n)]d_n \text{ for } d_n = 10, 20, 30, 40, 50
\]

- Constant costs and discount factor.

\[
\phi(d_n) = 0 \text{ for } d_n = 0
\]
DP SOLUTION APPROACH

<table>
<thead>
<tr>
<th>$d_n$</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi(d_n)$</td>
<td>0</td>
<td>48</td>
<td>86</td>
<td>118</td>
<td>138</td>
<td>160</td>
</tr>
</tbody>
</table>

- In the DP approach, at each stage $n$, we minimize the costs for the order in the stage $n$, $n-1$, $\ldots$, 1

$$f_n^*(s_n) = \min_{d_n} \left\{ \phi(d_n) + h_n [s_n + d_n - D_n] + f_{n-1}^*(s_{n-1}) \right\}$$

$$f_0(s_0) = 0 \text{ and so } f_0^*(s_0) = 0$$

DP SOLUTION: STAGE 1

$$s_0 = 0 \quad \text{and} \quad D_1 = 20 \Rightarrow s_1 = 20, 10 \text{ or } 0 \Rightarrow d_1^* = 0, 10 \text{ or } 20$$

$$f_1^*(s_1) = \min_{d_1} \left\{ \phi(d_1) + 0 \right\} = \phi(d_1^*)$$

<table>
<thead>
<tr>
<th>$s_1$</th>
<th>20</th>
<th>10</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_1^*$</td>
<td>0</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>$f_1^*(s_1)$</td>
<td>0</td>
<td>48</td>
<td>86</td>
</tr>
</tbody>
</table>
DP SOLUTION: STAGE 2

\[ s_1 = s_2 + d_2 - 30 \text{ since } D_2 = 30 \]
\[ f_2^*(s_2) = \min_{d_2} \left\{ \phi(d_2) + 0.2 \left[ s_2 + d_2 - 30 \right] + f_1^*(s_1) \right\} \]

<table>
<thead>
<tr>
<th>( s_2 )</th>
<th>( d_2 )</th>
<th>( d_2^* )</th>
<th>( f_2^*(s_2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10</td>
<td>20</td>
<td>30</td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
<td>204</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td>172</td>
</tr>
<tr>
<td>20</td>
<td>86</td>
<td>98</td>
<td>90</td>
</tr>
<tr>
<td>30</td>
<td>50</td>
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<td></td>
</tr>
<tr>
<td>40</td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

DP SOLUTION: STAGE 3

\[ s_2 = s_3 + d_3 - 40 \text{ since } D_3 = 40 \]
\[ f_3^*(s_3) = \min_{d_3} \left\{ \phi(d_3) + 0.2 \left[ s_3 + d_3 - 40 \right] + f_2^*(s_2) \right\} \]

<table>
<thead>
<tr>
<th>( s_3 )</th>
<th>( d_3 )</th>
<th>( d_3^* )</th>
<th>( f_3^*(s_3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10</td>
<td>20</td>
<td>30</td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
<td>302</td>
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<tr>
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<td></td>
<td></td>
<td>282</td>
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<tr>
<td>20</td>
<td>250</td>
<td>262</td>
<td>264</td>
</tr>
<tr>
<td>30</td>
<td>212</td>
<td>230</td>
<td>244</td>
</tr>
<tr>
<td>40</td>
<td>164</td>
<td>192</td>
<td>212</td>
</tr>
</tbody>
</table>
**DP SOLUTION: STAGE 4**

\[ s_3 = s_4 + d_4 - 30 \text{ since } D_4 = 30 \]

\[ f_4^*(s_4) = \min_{d_4} \left\{ \phi(d_4) + 0.2 \left[ s_4 + d_4 - 30 \right] + f_3^*(s_3) \right\} \]

<table>
<thead>
<tr>
<th>( s_4 )</th>
<th>( d_4 )</th>
<th>( d^*_4 )</th>
<th>( f_4^*(s_4) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>10</td>
<td>20</td>
<td>30</td>
<td>40</td>
</tr>
<tr>
<td>20</td>
<td>30</td>
<td>40</td>
<td>50</td>
</tr>
<tr>
<td>30</td>
<td>40</td>
<td>50</td>
<td>60</td>
</tr>
<tr>
<td>40</td>
<td>50</td>
<td>60</td>
<td>70</td>
</tr>
</tbody>
</table>

**DP SOLUTION: STAGE 5**

\[ s_4 = s_5 + d_5 - 20 \text{ since } D_5 = 20 \]

\[ f_5^*(s_5) = \min_{d_5} \left\{ \phi(d_5) + 0.2 \left[ s_5 + d_5 - 20 \right] + f_4^*(s_4) \right\} \]

<table>
<thead>
<tr>
<th>( s_5 )</th>
<th>( d_5 )</th>
<th>( d^*_5 )</th>
<th>( f_5^*(s_5) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>10</td>
<td>20</td>
<td>30</td>
<td>40</td>
</tr>
<tr>
<td>20</td>
<td>30</td>
<td>40</td>
<td>50</td>
</tr>
<tr>
<td>30</td>
<td>40</td>
<td>50</td>
<td>60</td>
</tr>
<tr>
<td>40</td>
<td>50</td>
<td>60</td>
<td>70</td>
</tr>
</tbody>
</table>
DP SOLUTION: STAGE 6

\[ D_6 = 40 \quad \text{and} \quad s_6 = 0 \]

\[ s_5 = s_6 + d_6 - 40 = d_6 - 40 \]

\[ f_6^*(s_6) = \min_{d_6} \left\{ \phi(d_6) + 0.2 \left[ \frac{s_6 + d_6 - 40}{s_5} \right] + f_5^*(s_5) \right\} \]

<table>
<thead>
<tr>
<th>( d_6 )</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>( d_6^* )</th>
<th>( f_6^*(s_6) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_6(s_6) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>606</td>
<td>608</td>
</tr>
</tbody>
</table>

\[ d_6^* = 40 \Rightarrow d_5^* = 50 \Rightarrow d_4^* = 0 \Rightarrow d_3^* = 40 \Rightarrow d_2^* = 50 \Rightarrow d_1^* = 0 \]

OPTIMAL SOLUTION

\[ d_6^* = 40 \quad \text{which implies} \quad s_5 = 0 \quad \text{and costs} \quad 606 \]

\[ d_5^* = 50 \quad \text{which implies} \quad s_4 = 30 \quad \text{and costs} \quad 468 \]

\[ d_4^* = 0 \quad \text{which implies} \quad s_3 = 0 \quad \text{and costs} \quad 302 \]

\[ d_3^* = 40 \quad \text{which implies} \quad s_2 = 0 \quad \text{and costs} \quad 302 \]

\[ d_2^* = 50 \quad \text{which implies} \quad s_1 = 20 \quad \text{and costs} \quad 164 \]

\[ d_1^* = 0 \quad \text{with costs} \quad 0 \]
The optimal trajectory is

\[ s_0 = 0 \rightarrow s_1 = 20 \rightarrow s_2 = 0 \rightarrow s_3 = 0 \rightarrow s_4 = 30 \rightarrow s_5 = 0 \]

The total costs for the sequence of orders are given by

\[ 0 + 164 + 138 + 0 + 166 + 138 = 606 \]
MUTUAL FUND INVESTMENT STRATEGIES

We consider a 5–year investment of

- 10 k$ invested in year 1
- 1 k$ invested in each year 2, 3, 4 and 5 into 2 mutual funds with different yields for both the short-term (1 year) and the long-term (up to 5 years)

The decision on the allocation of investment in each fund is made at the beginning of each year.

We operate under the following protocol:

- each fund returns short–term dividends and long–term dividends
- once invested, the money cannot be withdrawn until the end of the 5 – year period
- all short–term gains may either be reinvested in one of the two funds, or withdrawn; in the latter case, the withdrawn funds earn no further interest

Our objective is to maximize the total returns at the end of 5 years.
MUTUAL FUND INVESTMENT STRATEGIES

- The earnings on the investment are
  - \( LTD \): the long–term dividend specified as \%/year return on the accumulated capital
  - \( STD \): the short–term interest dividend returned as cash to the investor at the end of the period; cash may be invested in either fund and any money not invested earns no return

MUTUAL FUND INVESTMENT PARAMETERS

<table>
<thead>
<tr>
<th>fund</th>
<th>( STD ) rate ( i_n ) for year ( n )</th>
<th>( LTD ) rate ( I )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( A )</td>
<td>0.02</td>
<td>0.0225</td>
</tr>
<tr>
<td>( B )</td>
<td>0.06</td>
<td>0.0475</td>
</tr>
</tbody>
</table>
**DP SOLUTION APPROACH**

- We use backwards DP to solve the problem
- The stages are the 5 investment periods

\[ \text{stage } n \triangleq \text{year } 6 - n \quad n = 1, 2, 3, 4, 5 \]

**DP SOLUTION METHOD**

- For stage \( n \), the state \( s_n \) is the capital available for investment in the year \( 6 - n \)
- The decision \( d_n \) is the amount of capital invested in fund \( A \) in year \( 6 - n \) and so the amount of capital invested in fund \( B \) in the year \( 6 - n \) is \( s_n - d_n \)
- In each year, we determine the amount to invest in fund \( A \) and in fund \( B \) in order to optimize the returns at the end of year 5
DP SOLUTION METHOD

☐ The backward recursion application considers year 5 first and then each previous year in sequence

☐ Basic considerations:

☐ for each of the stages $6 - n$, $n = 1, \ldots, 5$,

$$d_n \text{ is invested in fund } A \text{ with returns } d_n i_A \ (STD)$$

and $(s_n - d_n)$ is invested in fund $B$ with returns

$$(s_n - d_n) i_B \ (STD)$$

☐ for the stage $6 - n + 1$, the $STD$s are augmented by $1,000$

$$s_{n-1} = d_n i_A + (s_n - d_n) i_B + 1,000 \quad n = 2, 3, 4, 5$$

☐ For the stage 5, we have the initial investment

$$s_5 = 10,000$$
THE OBJECTIVE

- The objective is to maximize the total returns
  \[ \max R = \sum_{n=1}^{5} r_n \]
  evaluated at the end of year 5
- We express all returns in end of the year 5 dollars:
  in each stage \( n \geq 2 \), \( r_n \) is the future value of long-term earnings in years 1, 2, 3 and 4, i.e., years \( 6 - n \)
  \[ r_n = (1 + I_A)^n d_n + (1 + I_B)^n (s_n - d_n) \quad n = 2, \ldots, 5 \]
- But for \( n = 1 \), \( r_1 \) is simply the present value of all earnings in stage 1, i.e., year 5
  \[ r_1 = (1 + I_A) d_1 + (1 + I_B) (s_1 - d_1) + i_A d_1 + i_B (s_1 - d_1) \]

DP SOLUTION: STAGE 1

- For stage 1

  \[ r_1 = (1 + I_A) d_1 + (1 + I_B) (s_1 - d_1) + i_A d_1 + i_B (s_1 - d_1) \]
  \[ r_1 = (I_A + i_A - I_B - i_B) d_1 + (1 + I_B + i_B) s_1 \]
**DP SOLUTION: STAGE 1**

- **$r_1$** = earnings in stage 1 (associated with the stage 1 decision)

\[
f_1^*(s_1) = \max_{d_1} \{ r_1 \} = \max_{d_1} \left\{ d_1 (I_A + i_{1A} - I_B - i_{1B}) + s_1 (1 + I_B + i_{1B}) \right\}
\]

\[
= \max_{0 \leq d_1 \leq s_1} \left\{ d_1 (0.04 + 0.025 - 0.03 - 0.04) + s_1 (1 + 0.03 + 0.04) \right\}
\]

optimal decision $d_1^* = 0$ with $f_1^*(s_1) = 1.07s_1$

- *maximum return in stage 1*

**DP SOLUTION: STAGE 2**

- **$r_2$** = returns associated with the decision in stage 2 realized at the end of year 5

\[
= d_2 (1 + I_A)^2 + (s_2 - d_2)(1 + I_B)^2
\]

\[
= d_2 \left[(1 + I_A)^2 - (1 + I_B)^2 \right] + s_2 (1 + I_B)^2
\]

- As a consequence of the decision $d_2$, the funds for investment in stage 1 are

\[
s_1 = s_2 i_{1B} + d_2 (i_{1A} - i_{1B}) + 1,000
\]
DP SOLUTION: STAGE 2

☐ We select $d^*_2$ to maximize

$$ f^*_2(s_2) = \max_{d_2} \left\{ r_2 + f^*_1(s_1) \right\} $$

$$ = \max_{0 \leq d_2 \leq s_2} \left\{ d_2 \left( .0207 \right) + 1.0609s_2 + 1.07\left[ .04s_2 + d_2 \left( - .015 \right) + 1,000 \right]\right\} $$

$$ = \max_{d_2} \left\{ d_2 \left( 1.04^2 - 1.03^2 \right) + s_2 \left( 1.03 \right)^2 + f^*_1(s_1) \right\} $$

$$ d^*_2 = s_2 \quad \text{with} \quad f^*_2(s_2) = 1.108s_2 + 1,070 $$

DP SOLUTION: STAGE 3

☐ $r_3 =$ returns associated with the decision $d_3$
realized at the end of 5 years

$$ = d_3 \left( 1 + I_A \right)^3 + (s_3 - d_3)(1 + I_B)^3 $$

$$ = d_3 \left[ (1 + I_A)^3 - (1 + I_B)^3 \right] + s_3 \left( 1 + I_B \right)^3 $$

☐ As a consequence of the decision $d_3$, the funds for investment in stage 2 are

$$ s_2 = s_3 i_{3B} + d_3 (i_{3A} - i_{3B}) + 1,000 $$
DP SOLUTION: STAGE 3

We select \( d^*_3 \) to maximize

\[
f_3^*(s_3) = \max_{d_3} \left\{ r_3 + f_2^*(s_2) \right\}
\]

\[
= \max_{d_3} \left\{ d_3 (1.04^3 - 1.03^3) + s_3 (1.03)^3 + \right\} \left\{ 1.108s_2 + 1,070 \right\}
\]

\[
= \max_{0 \leq d_3 \leq s_3} \left\{ 2,178 + 1.1481s_3 + .0018d_3 \right\}
\]

\[
d^*_3 = s_3 \quad \text{with} \quad f_3^*(s_3) = 1.15s_3 + 2,178
\]

DP SOLUTION: STAGE 4

\( r_4 = \) returns associated with the decision \( d_4 \)

realized at the end of 5 years

\[
= d_4 (1 + I_A)^4 + (s_4 - d_4)(1 + I_B)^4
\]

\[
= d_4 \left[ (1 + I_A)^4 - (1 + I_B)^4 \right] + s_4 (1 + I_B)^4
\]

The funds for investment in stage 3 depend explicitly on \( d_4 \)

\[
s_3 = s_4 i_{4R} + d_4 (i_{4A} - i_{4R}) + 1,000
\]
**DP SOLUTION: STAGE 4**

- We select $d_4^*$ to maximize

$$f_4^*(s_4) = \max_{d_4} \left\{ r_4 + f_3^*(s_3) \right\}$$

$$= \max_{d_4} \left\{ d_4 (1.04^4 - 1.03^4) + s_4 (1.03)^4 + 1.15 s_3 + 2,178 \right\}$$

$$= \max_{0 \leq d_4 \leq s_4} \left\{ 3328 + 1.1772 s_4 + .0156 d_4 \right\}$$

$$d_4^* = s_4 \quad \text{with} \quad f_4^*(s_4) = 1.193 s_4 + 3,328$$

**DP SOLUTION: STAGE 5**

- $r_5 = \text{returns associated with the decision } d_5$ realized at the end of 5 years

$$= d_5 (1 + I_A)^5 + (s_5 - d_5) (1 + I_B)^5$$

$$= d_5 \left[ 1.04^5 - 1.03^5 \right] + s_5 (1.03)^5$$

- The funds available in stage 5 are

$s_5 = 10,000$

- Therefore, the funds available for investment in stage 4 are
**DP SOLUTION: STAGE 5**

\[
s_4 = s_5 i_B + d_5 (i_A - i_B) + 1,000
\]

\[
= 10,000 i_B + d_5 (i_A - i_B) + 1,000
\]

We select \( d_5^* \) to maximize

\[
f^*_5(s_5) = \max_{0 \leq d_5 \leq s_5} \left\{ \frac{10,000(1.03)^5}{11.593} + d_5 (1.04^5 - 1.03^5) + f^*_4(s_4) \right\}
\]

\[
1,000 + 600 + d_5 (-0.04)
\]

\[
1.193 + 3,328
\]

\[
d_5^* = 10,000 \quad \text{with} \quad f^*_5(s_5) = 16,927
\]
**OPTIMAL SOLUTION**

*optimal* return at end of 5 years is 16,927 using the following strategy

<table>
<thead>
<tr>
<th>beginning of year</th>
<th>investment in fund A</th>
<th>investment in fund B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10,000</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>STD returns + 1,000</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>STD returns + 1,000</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>STD returns + 1,000</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>STD returns + 1,000</td>
</tr>
</tbody>
</table>