ECE 307 – Techniques for Engineering Decisions

Lecture 7. Dynamic Programming

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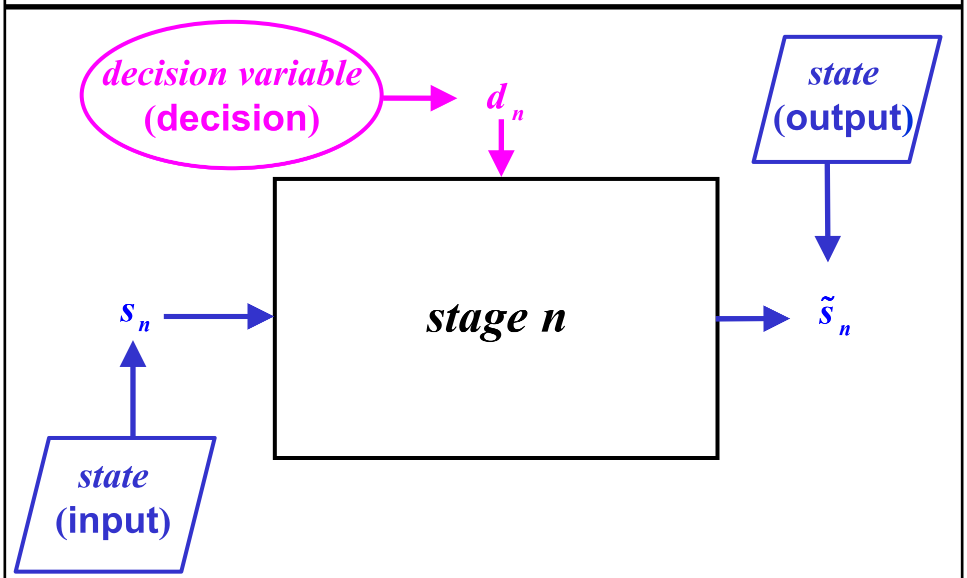
DYNAMIC PROGRAMMING

- Systematic approach to solving *sequential decision making* problems
- Salient problem characteristic: ability to *separate* the problem into *stages*
- *Multi-stage* problem solving technique

STAGES AND *STATES*

- We consider the problem to consist of *multiple separable stages*
- A *stage* may be a "point" in time, or space,or geography, or a structural element at which we make a decision; each *stage* is associated with one or more *states*
- A *state* of the system describes a possible configuration of the system in a given *stage*

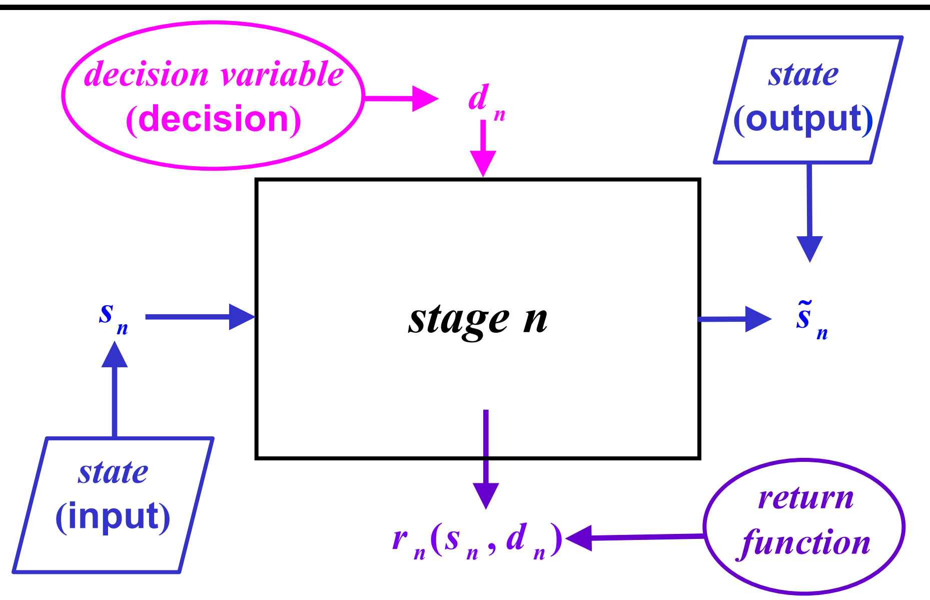
STAGES AND *STATES*



RETURN FUNCTION

- A *decision* d_n in the *stage n* *transforms* the *state* s_n in the *stage n* into the *state* s_{n+1} in the *stage* $n+1$
- The *state* s_n and the *decision* d_n have an impact on the objective function; the effect is measured in terms of the *return function* denoted by $r_n(s_n, d_n)$
- The *optimal* decision at *stage* n is the *decision* d_n^* that optimizes the *return function* for the *state* s_n

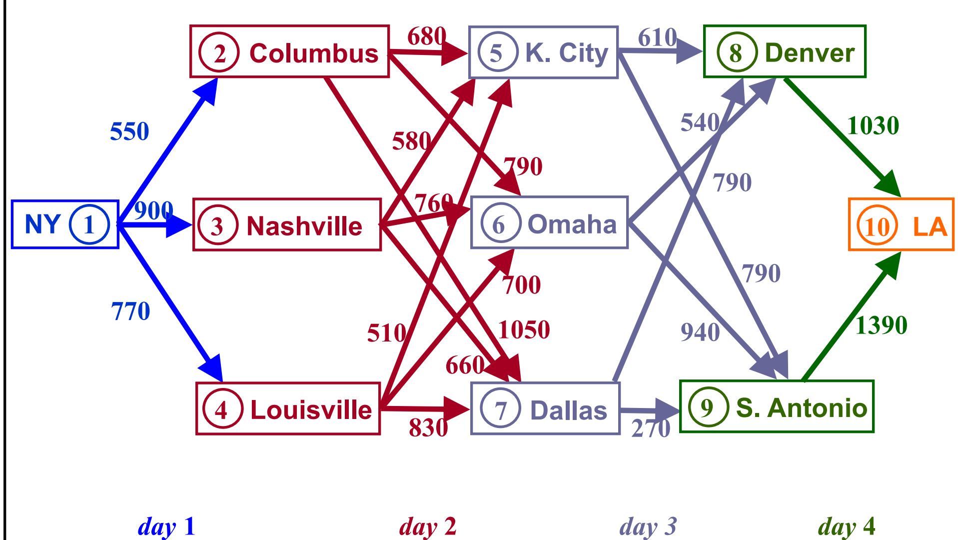
RETURN FUNCTION



ROAD TRIP EXAMPLE

- A poor student is traveling from NY to LA
- To minimize costs, the student plans to sleep at friends' houses each night in cities along the trip
- Based on past experience he can reach
  - Columbus, Nashville or Louisville after 1 day
  - Kansas City, Omaha or Dallas after 2 days
  - San Antonio or Denver after 3 days
  - LA after 4 days

ROAD TRIP EXAMPLE



ROAD TRIP

- The student wishes to minimize the number of miles driven and so he wishes to determine the *shortest path* from NY to LA
- To solve the problem, he works *backwards*
- We adopt the following notation

$c_{i,j}$ = distance between *states* i and j

$f_k(i)$ = distance of the shortest path to LA from *state* i in the *stage* k

ROAD TRIP EXAMPLE CALCULATIONS

day 4 : $f_4(8) = c_{8,10} = 1{,}030 \qquad f_4(9) = c_{9,10} = 1{,}390$

day 3 : $$f_3(5) = min\left\{\underbrace{(610+1{,}030)}_{1{,}640},\underbrace{(790+1{,}390)}_{2{,}180}\right\} = 1{,}640$$

$$f_3(6) = min\left\{\underbrace{(540+1{,}030)}_{1{,}570},\underbrace{(940+1{,}390)}_{2{,}330}\right\} = 1{,}570$$

$$f_3(7) = min\left\{\underbrace{(790+1{,}030)}_{1{,}820},\underbrace{(270+1{,}390)}_{1{,}660}\right\} = 1{,}660$$

ROAD TRIP EXAMPLE CALCULATIONS

day 2 :

$$f_2(2) = min \left\{ \underbrace{(680+1,640)}_{2,320}, \underbrace{(790+1,570)}_{2,360}, \underbrace{(1,050+1,660)}_{2,710} \right\} = 2,320$$

$$f_2(3) = min \left\{ \underbrace{(580+1,640)}_{2,220}, \underbrace{(760+1,570)}_{2,330}, \underbrace{(660+1,660)}_{2,320} \right\} = 2,220$$

$$f_2(4) = min \left\{ \underbrace{(510+1,640)}_{2,150}, \underbrace{(700+1,570)}_{2,270}, \underbrace{(830+1,660)}_{2,490} \right\} = 2,150$$

ROAD TRIP EXAMPLE

day 1 :

$$f_1(1) = min \left\{ \underbrace{(550+2,320)}_{*2,870*}, \underbrace{(900+2,220)}_{3,120}, \underbrace{(770+2,150)}_{2,920} \right\} = 2,870$$

- **The shortest path is 2,870 miles and corresponds to the trajectory $\{ (1,2), (2,5), (5,8), (8,10) \}$, *i.e.*, from NY, the student reaches Columbus on the first day, Kansas City on the second day, Denver the third day and then LA**
- **Every other trajectory to LA leads to higher costs and so is, by definition, *suboptimal***

PICK UP MATCHES GAME

- There are 30 matches on a table and 2 players
- Each player can pick up 1, 2, or 3 matches and continue until the last match is picked up
- The loser is the person who picks up the *last match*
- How can the player P_1, who goes first, ensure to be the winner?

WORKING BACKWARDS: *PICK UP MATCHES* GAME

- We solve this problem by reasoning in a back-wards fashion so as to ensure that when a single match remains, P_2 has the turn
- Consider the situation where 5 matches remain and it is P_2' s turn; for P_1 to win, we consider all possible situations:

WORKING BACKWARDS: *PICK UP MATCHES* GAME

P_2's move is to pick

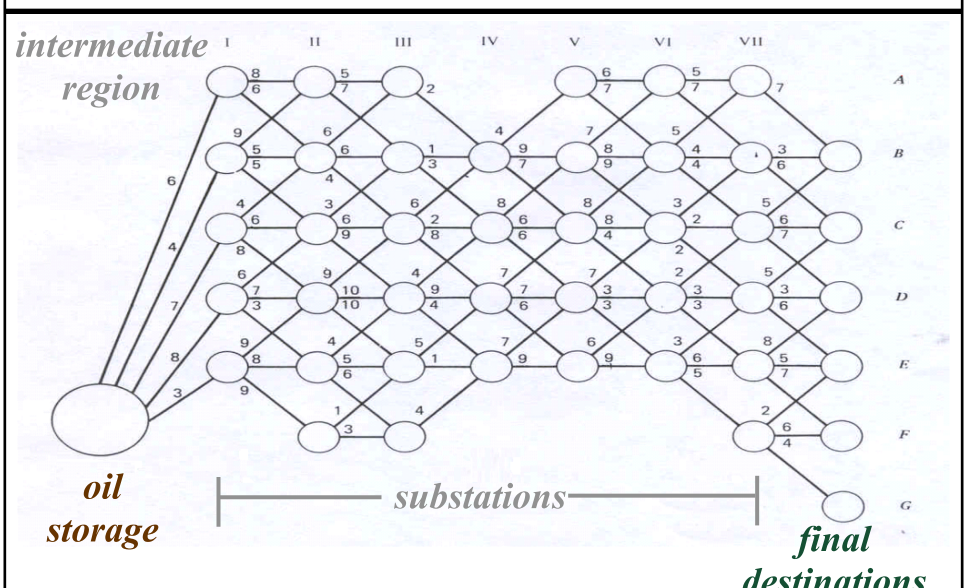
- $1 \Rightarrow 4$ left ➔ P_1 removes 3
- $2 \Rightarrow 3$ left ➔ P_1 removes 2
- $3 \Rightarrow 2$ left ➔ P_1 removes 1

❑ We can reason similarly for the cases of $9, 13, 17, 21, 25$, and 29 matches

❑ Therefore, P_1 wins if P_1 picks $30 - 29 = 1$ match in the first move

❑ In this manner, we can assure a win for any number of matches in the game

OIL TRANSPORT TECHNOLOGY



OIL TRANSPORT TECHNOLOGY

- We consider the development of a transport network from the north slope of Alaska to one of 6 possible shipping points in the *US*
- The network must meet the problem feasibility requirements:
  - 7 pumping stations from a north slope ground storage plant to a shipping port

OIL TRANSPORT TECHNOLOGY

  - use of only those paths that are physically and environmentally feasible
- Objective: determine a *feasible* pumping configuration that minimizes the

$$\text{total costs} = \sum \text{construction costs of branches of network of the feasible pumping configuration}$$

OIL TRANSPORT TECHNOLOGY

- **Possible approaches to solving such a problem include:**
  - *enumeration*: **exhaustive evaluation of all possible paths, which is too costly since there are more than 100 possible paths for this small size problem**
  - *myopic decision rule*: **at each node, pick as the next node the one reachable by the cheapest path (in case of ties the pick is arbitrary); we show a possible path**

OIL TRANSPORT TECHNOLOGY

0 →	3 →	11 →	15 →	19 →	25 →	29 →	31 →	36
oil storage	I-*E*	II-*E*	III-*D*	IV-*E*	V-*C*	VI-*D*	VII-*C*	*B*

but, such a path is not unique and cannot be guaranteed to be *optimal*

- *serial dynamic programming* (*DP*) **: we need to construct the problem solution by defining the** *stages*, *states* **and** *decisions*

DP SOLUTION

- We define an intermediate *stage* to represent each pumping region and so each such *stage* corresponds to the set of vertical nodes in regions $\mathrm{I}, \mathrm{II}, \ldots, \mathrm{VII}$
- We also define a stage of final destinations and the initial stage for oil storage
- We use *backwards recursion*: we start from every final destination and work *backwards* to the oil storage *stage*

DP SOLUTION

- We define a *state* s_k to denote a final destination, a specific pumping station in the intermediate regions or the oil storage tank with all the oil
- A decision refers to the selection of the branch from each *state* s_k, so there are at most three choices for a *decision* d_k:

$L \leftrightarrow$ left $\quad F \leftrightarrow$ forward $\quad R \leftrightarrow$ right

DP SOLUTION

- The *return function* $r_k(s_k, d_k)$ is defined as the costs associated with the *decision* d_k for the *state* s_k
- The transition function is the total costs in proceeding from a state s_{k+1} in *stage* $k+1$ to another state s_k in *stage* k , $k = 0, 1, \ldots, 7$
- We solve the problem by iteratively moving *backwards*, starting from each final *state* to the *states* in *stage* 1 and so on, until we reach the oil storage

DP SOLUTION: *STAGE* 1 ↔ REGION VII TO A FINAL DESTINATION

optimal decision

optimal return

s_1	d_1			d_1^*	$f_1^*(s_1)$
	R	*L*	*F*		
A	7			*R*	7
B	6		3	*F*	3
C	7	5	6	*L*	5
D	6	5	3	*F*	3
E	7	8	5	*F*	5
F	4	2	6	*L*	2

least costs in proceeding from the state s_1 to a final destination

DP SOLUTION: *STAGE* 2 ↔ REGION VI TO *STAGE* 1

optimal decision

s_2	d_2			d^*_2	$f^*_2(s_2)$
	R	*L*	*F*		
A	10		12	*R*	10
B	9	12	7	*F*	7
C	5	6	7	*R*	5
D	8	7	6	*F*	6
E	7	6	11	*L*	6

cumulative costs in proceeding from the state s_2 to a final destination

STAGE 2 CALCULATION

costs of proceeding from the state s_2 to a state s_1 in stage 1

$$\underbrace{f^*_2(s_2) = \min_{d_2} \left(r_2(s_2, d_2) + \underbrace{f^*_1(s_1)}_{} \right)}_{}$$

a function of only s_1

$\Downarrow$

for a given d_2, the state s_1 is set

DP SOLUTION: *STAGE* 3 ◂▸ REGION V TO *STAGE* 2

$$f^*_3(s_3) = \min_{d_3} \{ r_3(s_3, d_3) + f^*_2(s_2) \}$$

s_3	d_3			d^*_3	$f^*_3(s_3)$
	R	*L*	*F*		
A	14		16	*R*	14
B	14	17	15	*R*	14
C	10	5	13	*R*	10
D	9	12	9	*R, F*	9
E		12	15	*L*	12

cumulative costs in proceeding from the state s_3 to a final destination

DP SOLUTION: *STAGE* 4 ◂▸ REGION IV TO *STAGE* 3

$$f^*_4(s_4) = \min_{d_4} \{ r_4(s_4, d_4) + f^*_3(s_3) \}$$

s_4	d_4			d^*_4	$f^*_4(s_4)$
	R	*L*	*F*		
B	17	18	23	*R*	17
C	15	22	16	*R*	15
D	18	17	16	*F*	16
E		16	21	*L*	16

cumulative costs in proceeding from the state s_4 to a final destination

DP SOLUTION: *STAGE* 5 ◆ REGION III TO *STAGE* 4

$$f^*_5(s_5) = \min_{d_5} \left\{ r_5(s_5, d_5) + f^*_4(s_4) \right\}$$

s_5	d_5			d^*_5	$f^*_5(s_5)$
	R	*L*	*F*		
A	19			*R*	19
B	18		18	*R, F*	18
C	24	23	17	*F*	17
D	20	19	25	*L*	19
E		21	17	*F*	17
F		20		*L*	20

cumulative costs in proceeding from the state s_5 to a final destination

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DP SOLUTION: *STAGE* 6 ◆ REGION II TO *STAGE* 6

$$f^*_6(s_6) = \min_{d_6} \left\{ r_6(s_6, d_6) + f^*_5(s_5) \right\}$$

s_6	d_6			d^*_6	$f^*_6(s_6)$
	R	*L*	*F*		
A	25		24	*F*	24
B	21	25	24	*R*	21
C	28	21	23	*L*	21
D	27	26	29	*L*	26
E	26	23	22	*F*	22
F		18	23	*L*	18

cumulative costs in proceeding from the state s_6 to a final destination

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DP SOLUTION: *STAGE* 6 ◄► REGION II TO *STAGE* 6

$$f^*_7(s_7) = \min_{d_7} \left\{ r_7(s_7, d_7) + f^*_6(s_6) \right\}$$

s_7	d_7			d^*_7	$f^*_7(s_7)$
	R	L	F		
A	27		32	R	27
B	26	33	26	R,F	26
C	34	25	27	L	25
D	25	27	33	R	25
E	27	35	30	R	27

cumulative costs in proceeding from the state s_7 to a final destination

THE *OPTIMAL* TRAJECTORY

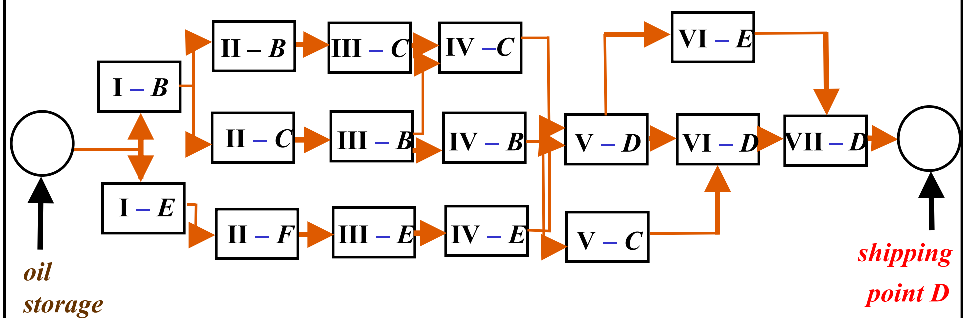
- The last *stage* consists of only 1 *state* – the oil storage

s_8 \ d_8	A	B	C	D	E	d^*_8	$f^*_8(s_8)$
$f_8(s_8)$	33	30	32	33	30	B,E	30

$$f^*_8(s_8) = \min\left\{27+6, 26+4, 25+7, 25+8, 27+3\right\} = 30$$

- To find the *optimal* trajectory, we retrace in the forward direction and go through the *stages* 7, 6, 5, . . . , 1 to get the *least–cost trajectory* that terminates in shipping point *D*

THE *OPTIMAL* TRAJECTORY

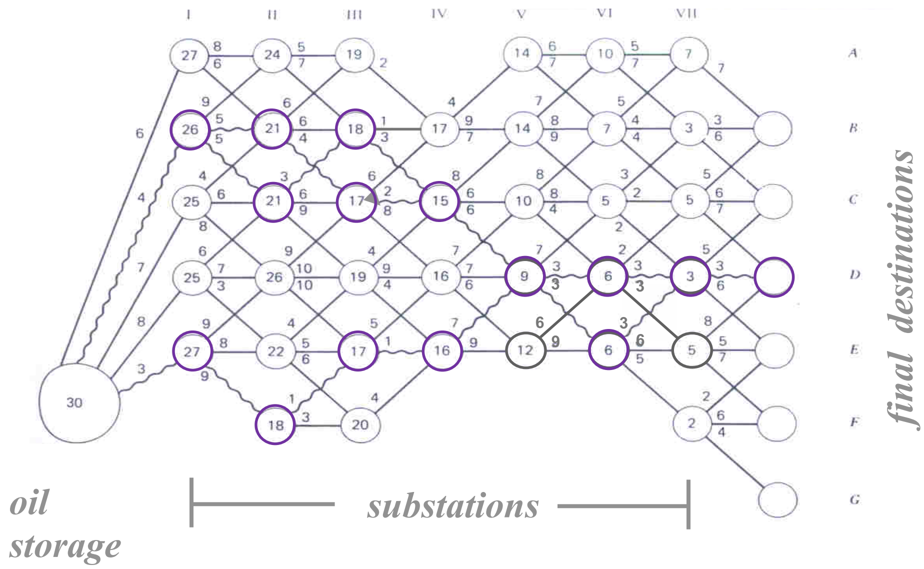


- Besides this *optimal* solution, other trajectories are possible since the path need not be unique but no path yields a shorter total distance

OIL TRANSPORT PROBLEM SOLUTION

- We obtain the diagram below by retracing the steps of proceeding to an endpoint at each *stage*
- The solution
  - provides all the *optimal trajectories*
  - is based on logically breaking up the problem into *stages* with calculations in each *stage* being a function of the number of *states* in that *stage*
  - provides also all the *suboptimal paths*

OIL TRANSPORT PROBLEM SOLUTION



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OIL TRANSPORT PROBLEM SOLUTION

- For example, we may calculate the least cost *optimal* path to any ***sub–optimal*** shipping point different than ***D***
- From the solution, we can also determine the ***sub–optimal*** path if the construction of a feasible path is not undertaken

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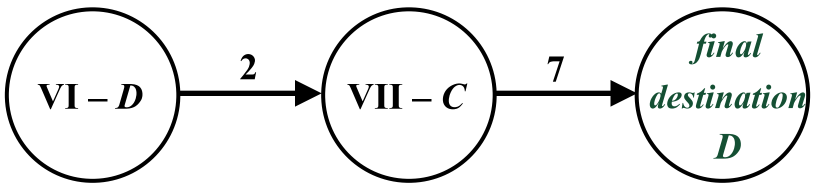
OIL TRANSPORT: SENSITIVITY CASE

- Consider the case where we got to *stage* VI but the branch VI – *D* to VII – *D* cannot be built due to some newly–enacted environmental constraint
- We then determine the least–cost path from VI – *D* to find the final destination *D* whose value is 9 instead of 6



and so the sub *optimal* cost solution costs are 33

FACILITIES SELECTION PROBLEM

- A company is expanding to meet a wider market and considers:
  - 3 location alternatives
  - 4 different building types (sizes) at each site
- Revenues – meaning *net revenues* or *profits* – and costs vary with each location and building type

FACILITIES SELECTION PROBLEM

- Revenues R increase monotonically with building size
- Costs C also increase monotonically with building size
- The data for building sizes and the associated revenues and costs are given in the table below

FACILITIES SELECTION PROBLEM

site	building size									
	B_1		B_2		B_3		B_4		none	
	R_1	c_1	R_2	c_2	R_3	c_3	R_4	c_4	R_0	c_0
I	0.50	1	0.65	2	0.8	3	1.4	5	0	0
II	0.62	2	0.78	5	0.96	6	1.8	8	0	0
III	0.71	4	1.2	7	1.6	9	2	11	0	0

FACILITIES SELECTION PROBLEM

- **The company investment budget is limited to *$ 21 million* for the total expansion project**
- **The goal is to determine the *optimal* expansion policy, *i.e.*, the buildings to be built at each site**

DP SOLUTION APPROACH

- **We use the *DP* approach to solve this problem, but first, we must define the *DP* structure elements**
- **For the facilities siting problem, we realize that absent the choice of a site, the building type is irrelevant and so the elements that control the entire decision process are the *building sites***

DP SOLUTION APPROACH

stage	$\leftrightarrow$	site
state	$\leftrightarrow$	amount of funds available for construction
decision	$\leftrightarrow$	building type
return function	$\leftrightarrow$	revenues
transition function	$\leftrightarrow$	impact of a decision on the available funds

DP SOLUTION APPROACH

- We use backwards *DP* to solve the problem and start with site I $\leftrightarrow$ *stage* 1 , a purely arbitrary choice, where this *stage* 1 represents the last decision in the 3 – *stage* sequence and so is made after the decision for the other two sites have been taken

DP SOLUTION APPROACH

- The amount of funds available is unknown since the decision at sites II and III are already made, and so

$$0 \leq s_1 \leq 21$$

- There are no additional decisions to be made in *stage 0* and we define

$$s_0 = 0 \quad \text{and} \quad f^*_0(s_0) = 0$$

DP SOLUTION APPROACH

- We start with *stage* 1 and move backwards to *stages* 2 and 3
- As we move backwards from *stage* $(n - 1)$ to *stage n*, as a result of the *decision* d_n, the funds available for construction in *stage* $(n - 1)$ are

$$s_{n-1} = s_n - c_n \leftarrow \text{costs of } decision\ d_n$$

DP SOLUTION APPROACH

- The recursion relation is given by

$$f_n^*(s_n) = \max_{d_n} \left\{ f_n(s_n, d_n) + f_{n-1}^*(s_{n-1}) \right\}, \quad n = 1,2,3$$

with

$$s_{n-1} = s_n - c_n$$

and

revenues for *decision* d_n

$$f_n(s_n, d_n) = r_n(s_n, d_n) = R_n$$

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DP SOLUTION: *STAGE* 1 ◂▸ SITE I

$$f_1^*(s_1) = \max_{0 \leq d_1 \leq 4} \underbrace{\{ r_1(s_1, d_1) \}}_{R_1}$$

s_1 \ d_1	*0*	1	2	3	4	d_1^*	$f_1^*(s_1)$
$21 \geq s_1 \geq 5$	*0*	.50	.65	.80	1.40	4	1.40
$4 \geq s_1 \geq 3$	*0*	.50	.65	.80		3	.80
2	*0*	.50	.65			2	.65
1	*0*	.50				1	.50
0	*0*	*0*	*0*	*0*	*0*	*0*	*0*

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DP SOLUTION: *STAGE* 2 ↔ SITE II

- The amount of funds s_2 available is unknown since the decision at site III is already made
- The value of d_2 is a function of s_2 and we construct a decision table using

$$f^*_2(s_2) = \max_{0 \leq d_2 \leq 4} \{\underbrace{r_2(s_2, d_2)}_{R_2} + f^*_1(s_1)\}$$

where

$$s_1 = s_2 - c_2$$

DP SOLUTION: *STAGE* 2 ↔ SITE II

s_2 \ d_2	*0*	1	2	3	4	d^*_2	$f^*_2(s_2)$
$21 \geq s_2 \geq 13$	1.40	2.02	2.18	2.36	3.20	4	3.20
12	1.40	2.02	2.18	2.36	2.60	4	2.60
11	1.40	2.02	2.18	2.36	2.60	4	2.60
10	1.40	2.02	2.18	1.76	2.45	4	2.45
9	1.40	2.02	1.58	1.61	2.30	4	2.30
8	1.40	2.02	1.58	1.61	1.80	1	2.02
7	1.40	2.02	1.43	1.46		1	2.02
6	1.40	1.42	1.28	0.96		1	1.42
5	1.40	1.42	0.78			1	1.42
4	0.80	1.27				1	1.27
3	0.80	1.12				1	1.12
2	0.65	0.62				*0*	0.65
1	0.50					*0*	0.50
0	0.00					*0*	0.00

SAMPLE CALCULATIONS

- Consider the case $s_2 = 10$ and $d_2 = 0$; then,

$$c_2 = 0 \quad \text{and} \quad R_2 = 0$$

and so,

$$s_1 = 10 \quad \text{and} \quad d_1^* = 4$$

- Therefore,

$$f_1^*(s_1) = 1.4$$

and consequently,

$$f_2(s_2) = 1.4$$

SAMPLE CALCULATIONS

- Consider next the case $s_2 = 10$ and $d_2 = 4$; then,

$$c_2 = 8 \quad \text{and} \quad R_2 = 1.8$$

and so,

$$s_1 = 2$$

so that

$$f_1^*(s_1) = 0.65$$

- Consequently,

$$f_2(s_2) = 2.45$$

which we can show is the optimal value, so that

$$f_2^*(s_2) = 2.45$$

DP SOLUTION : *STAGE* 3 ↔ SITE III

- At *stage* 3 , the first decision is actually taken and so exactly 21 million is available and $s_3 = 21$
- We compute the elements in the table using

$$f^*_3(s_3) = \max_{d_3}\{\underbrace{r_3(s_3, d_3)}_{R_3} + f^*_2(s_2)\}$$

where

$$s_2 = s_3 - c_3$$

OPTIMAL SOLUTION

s_3	d_3					d^*_3	$f^*_3(s_3)$
	0	1	2	3	4		
21	3.20	3.91	4.40	4.20	4.45	4	4.45

- *Optimal* profits are 4.45 million and the *optimal* path is obtained by retracing the steps from *stage* 3 to *stage* 1 in the forward direction:

OPTIMAL SOLUTION

$d_3^* = 4 \leftrightarrow$ construct B_4 at site III

$s_2 = s_3 - c_3 = 21 - 11 = 10$

$d_2^* = 4 \leftrightarrow$ construct B_4 at site II

$s_1 = s_2 - c_2 = 10 - 8 = 2$

$d_1^* = 2 \leftrightarrow$ construct B_2 at site I

$c_1 = 5$ and $c_1 + c_2 + c_3 = 21$

A SENSITIVITY CASE

- We next consider the case where the maximum investment available is 15 million
- By inspection, the results in *stages* 1 and 2 remain unchanged; however, we must recompute *stage* 3 results with the 15 million limit

s_3	d_3					d_3^*	$f_3^*(s_3)$
	0	1	2	3	4		
15	3.2	3.31	3.22	3.02	3.27	1	3.31

SENSITIVITY CASE

- The *optimal* solution obtains maximum profits of 3.31 million and the decision is as follows:

$d_3^* = 1 \quad \leftrightarrow$ construct B_1 at site III

$s_2 = s_3 - c_3 = 15 - 4 = 11$

$d_2^* = 4 \quad \leftrightarrow$ construct B_4 at site II

$s_1 = s_2 - c_2 = 11 - 8 = 3$

$d_1^* = 3 \quad \leftrightarrow$ construct B_3 at site I

$c_1 = 3$ and $c_1 + c_2 + c_3 = 15$