# ECE 307 – Techniques for Engineering Decisions

## Lecture 5. Networks and Flows

**George Gross**

Department of Electrical and Computer Engineering

University of Illinois at Urbana-Champaign

---

# NETWORKS AND FLOWS

- A network is a system of lines or channels or branches that connect different points
- Examples abound virtually in all aspects of life:
  - electrical systems;
  - communication networks;
  - airline webs;
  - local area networks; and
  - distribution systems

## NETWORKS AND FLOWS

- The network structure is also common to many other systems that at first glance are not necessarily viewed as networks
  - **distribution of products** through a system consisting of manufacturing plants, warehouses and retail outlets
  - **matching problems** such as work to people, tasks to machines and computer dating

## NETWORKS AND FLOWS

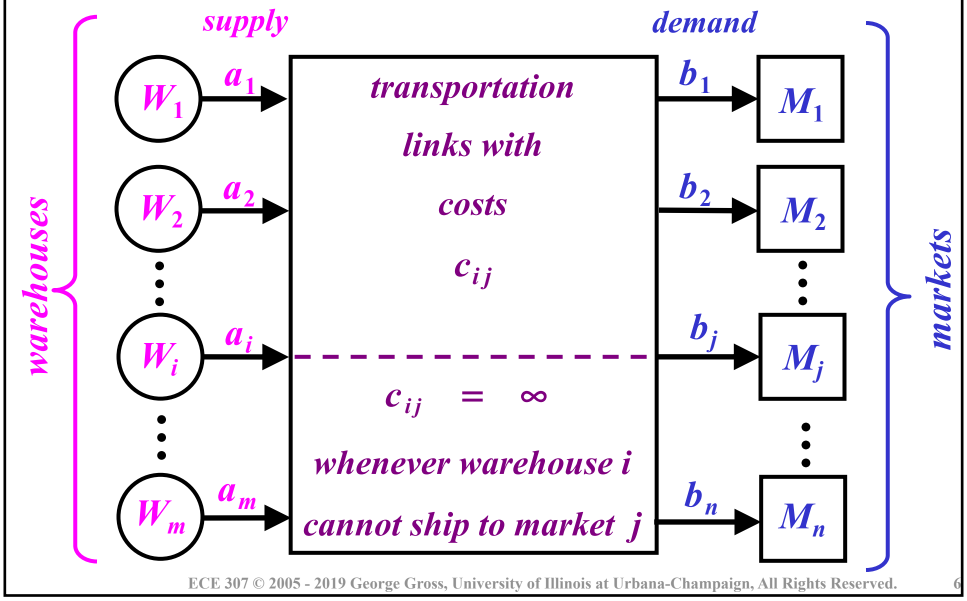
  - **river systems** with pondage for electricity generation
  - **mail delivery** networks
  - **freight delivery** networks
  - **project management** of multiple tasks in a large undertaking such as a major construction project or a space flight
- We consider a broad range of network and network flow problems

# THE TRANSPORTATION PROBLEM

- ☐ The basic idea of the transportation problem is illustrated with the problem of the distribution of a specified *homogeneous* product from several warehouses to a number of localities *at least cost*
- ☐ We consider a system with $m$ warehouses, $n$ markets and links between them with the specified costs of transportation

---

# THE TRANSPORTATION PROBLEM



*supply*

*demand*

*warehouses*

*markets*

$W_1$, $W_2$, $W_i$, $W_m$

$a_1$, $a_2$, $a_i$, $a_m$

*transportation links with costs* $c_{ij}$

$c_{ij} = \infty$

*whenever warehouse i cannot ship to market j*

$b_1$, $b_2$, $b_j$, $b_n$

$M_1$, $M_2$, $M_j$, $M_n$

## THE TRANSPORTATION PROBLEM

- all the supply comes from the $m$ warehouses; we associate the index $i = 1, 2, \dots, m$ with a warehouse
- all the demand is at the $n$ markets; we use the index $j = 1, 2, \dots, n$ with a market
- shipping costs $c_{ij}$ for each unit from the warehouse $i$ to the market $j$ and we set $c_{ij} = \infty$ whenever warehouse $i$ cannot ship to market $j$

## THE TRANSPORTATION PROBLEM

- The transportation problem is to determine the *optimal shipping schedule* that minimizes shipping costs from the set of $m$ warehouses to the set of $n$ markets by determining the quantities shipped from each warehouse $i$ to each market $j$, for $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$

## *LP* FORMULATION OF THE TRANSPORTATION PROBLEM

- ☐ The decision variables are defined to be

$x_{ij}$ = *quantity shipped from warehouse i to market j,*

$$i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n$$

- ☐ The objective function is

$$\min \quad \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

---

## *LP* FORMULATION OF THE TRANSPORTATION PROBLEM

- ☐ The constraints are:

$$\sum_{j=1}^{n} x_{ij} \leq a_i \qquad i = 1, 2, \dots, m$$

$$\sum_{i=1}^{m} x_{ij} \geq b_j \qquad j = 1, 2, \dots, n$$

$$x_{ij} \geq 0 \qquad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n$$

## *LP* FORMULATION OF THE TRANSPORTATION PROBLEM

- ❑ Note that feasibility requires that

$$\sum_{i=1}^{m} a_i \geq \sum_{j=1}^{n} b_j$$

- ❑ When

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

all the available supply at the $m$ warehouses is shipped to meet all the demands of the $n$ markets; this is known as the *standard transportation problem*

---

## STANDARD TRANSPORTATION PROBLEM (*STP*)

$$min \quad \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \, x_{ij}$$

$$s.t.$$

$$\left.\begin{array}{rcl} \sum_{j=1}^{n} x_{ij} & = & a_i \\ \sum_{i=1}^{m} x_{ij} & = & b_j \\ x_{ij} & \geq & 0 \end{array}\right\} \begin{array}{l} i = 1, \ldots, m \\ j = 1, \ldots, n \end{array}$$

# STANDARD TRANSPORTATION PROBLEM ($STP$)

- The standard transportation problem has
  - $m\,n$ variables $x_{ij}$
  - $m + n$ equality constraints
- However, since

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j ,$$

there are at most $(m + n - 1)$ *independent constraints* and, consequently, at most $(m + n - 1)$ *independent variables* $x_{ij}$ (*basic variables*)

# TRANSPORTATION PROBLEM SETUP

| *market j*<br>*w/h i* | $M_1$ | $M_2$ | $M_3$ | $M_4$ | *supplies* |
|---|---|---|---|---|---|
| $W_1$ | $x_{11}$<br>$c_{11}$ | $x_{12}$<br>$c_{12}$ | $x_{13}$<br>$c_{13}$ | $x_{14}$<br>$c_{14}$ | $a_1$ |
| $W_2$ | $x_{21}$<br>$c_{21}$ | $x_{22}$<br>$c_{22}$ | $x_{23}$<br>$c_{23}$ | $x_{24}$<br>$c_{24}$ | $a_2$ |
| $W_3$ | $x_{31}$<br>$c_{31}$ | $x_{32}$<br>$c_{32}$ | $x_{33}$<br>$c_{33}$ | $x_{34}$<br>$c_{34}$ | $a_3$ |
| *demands* | $b_1$ | $b_2$ | $b_3$ | $b_4$ | $\sum_i a_i = \sum_j b_j$ |

# TRANSPORTATION PROBLEM NUMERICAL EXAMPLE

| *market j*<br>*w/h i* | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $a_i$ |
|---|---|---|---|---|---|
| $W_1$ | 2 | 2 | 2 | 1 | 3 |
| $W_2$ | 10 | 8 | 5 | 4 | 7 |
| $W_3$ | 7 | 6 | 6 | 8 | 5 |
| $b_j$ | 4 | 3 | 4 | 4 | |

# THE LEAST – COST RULE PROCEDURE

- ❑ The *LCRP* generates an initial *basic feasible solution* which has at most $(m + n - 1)$ positive–valued *basic variables*
- ❑ The principal idea of the scheme is to select, at each step, the variable $x_{ij}$ with the *lowest shipping costs* $c_{ij}$ as the next *basic variable* to enter the basis

# APPLICATION OF THE LEAST – COST RULE

- ❑ $c_{14}$ is the lowest $c_{ij}$ and we select $x_{14}$ as a *basic variable*
- ❑ We choose $x_{14}$ as large as possible without violating any constraints:

$$min \ \{ a_1 , b_4 \} \ = \ min \ \{ 3 , 4 \} \ = \ 3$$

- ❑ We set $x_{14} = 3$ and set

$$x_{11} = x_{12} = x_{13} = 0$$

- ❑ We delete row 1 from any further consideration since all the supplies from $W_1$ are exhausted

---

# APPLICATION OF THE LEAST – COST RULE

| market $j$ / w/h $i$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $a_i$ |
|---|---|---|---|---|---|
| $W_1$ | 2 | 2 | 2 | ③<br>1 | 3 |
| $W_2$ | 10 | 8 | 5 | 4 | 7 |
| $W_3$ | 7 | 6 | 6 | 8 | 5 |
| $b_j$ | 4 | 3 | 4 | 4 | |

# APPLICATION OF THE LEAST – COST RULE

- The remaining demand at $M_4$ is

$$4 - 3 = 1$$

which is the value for the modified demand at $M_4$

- We again apply the *criterion selection* to the reduced tableau: since $c_{24}$ is the lowest–valued $c_{ij}$ , we select $x_{24}$ as the next *basic variable*

# APPLICATION OF THE LEAST – COST RULE

- We wish to set $x_{24}$ as large as possible without violating any constraints:

$$min \; \{ a_2 , b_4 \} = min \; \{ 7 , 1 \} = 1$$

and we set $x_{24} = 1$ and since there is no more demand at $M_4$

$$x_{34} = 0$$

- We delete column 4 from any further consideration since all the demand at $M_4$ is met

# APPLICATION OF THE LEAST – COST RULE

- The remaining supply at $W_2$ is

$$7 - 1 = 6 ,$$

which is the value for the modified supply at $W_2$

- We repeat these steps until we find the values of the $(m + n - 1)$ nonzero *basic variables* to obtain a *basic feasible solution*

- In the reduced tableau,

---

# APPLICATION OF THE LEAST – COST RULE

| market *j* <br> w/h *i* | $M_1$ | $M_2$ | $M_3$ | $a_i$ |
|---|---|---|---|---|
| $W_2$ | $\boxed{10}$ | $\boxed{8}$ | ④ $\boxed{5}$ | 6 |
| $W_3$ | $\boxed{7}$ | $\boxed{6}$ | *0* $\boxed{6}$ | 5 |
| $b_j$ | 4 | 3 | 4 | |

# APPLICATION OF THE LEAST – COST RULE

- pick $x_{23}$ to enter the basis as the next basic variable
- set

$$x_{23} = min \{ 6, 4 \} = 4$$

  and set

$$x_{33} = 0$$

- eliminate column 3 and reduce the supply at $W_2$ to $6 - 4 = 2$

☐ For the reduced tableau

---

# APPLICATION OF THE LEAST – COST RULE

| *market j*<br>*w/h i* | $M_1$ | $M_2$ | $a_i$ |
|---|---|---|---|
| $W_2$ | 10 | *0*<br>8 | 2 |
| $W_3$ | 7 | ③<br>6 | 5 |
| $b_j$ | 4 | 3 | |

# APPLICATION OF THE LEAST – COST RULE

  - pick $x_{32}$ to enter the basis
  - set

$$x_{32} = min\{3, 5\} = 3$$

    and set

$$x_{22} = 0$$

  - eliminate column 2 in the reduced tableau and reduce the supply at $W_3$ to $5 - 3 = 2$

- The last reduced tableau is

---

# APPLICATION OF THE LEAST – COST RULE

| *market j*<br>*w/h i* | $M_1$ | $a_i$ |
|---|---|---|
| $W_2$ | $\boxed{10}$ | 2 |
| $W_3$ | ② $\boxed{7}$ | 2 |
| $b_j$ | 4 | |

# APPLICATION OF THE LEAST – COST RULE

- pick $x_{31}$ to enter the basis
- set

$$x_{31} = min \{ 2, 4 \} = 2$$

- reduce the demand at $M_1$ to

$$4 - 2 = 2$$

- the value of

$$x_{21} = 2$$

is obtained by default

---

# INITIAL *BASIC FEASIBLE SOLUTION*

| market $j$<br>w/h $i$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $a_i$ |
|---|---|---|---|---|---|
| $W_1$ | $\boxed{2}$ | $\boxed{2}$ | $\boxed{2}$ | 3<br>$\boxed{1}$ | 3 |
| $W_2$ | 2<br>$\boxed{10}$ | $\boxed{8}$ | 4<br>$\boxed{5}$ | 1<br>$\boxed{4}$ | 7 |
| $W_3$ | 2<br>$\boxed{7}$ | 3<br>$\boxed{6}$ | $\boxed{6}$ | $\boxed{8}$ | 5 |
| $b_j$ | 4 | 3 | 4 | 4 | |

# APPLICATION OF THE LEAST – COST RULE

- ☐ The feasible solution involves only the basic variables and results in shipment costs of

$$\sum_{i=1}^{3} \sum_{j=1}^{4} c_{ij}\, x_{ij} = 1\cdot 3 + 4\cdot 1 + 5\cdot 4 + 6\cdot 3 + 7\cdot 2 + 10\cdot 2$$

$$= 79$$

---

# THE *STP*

- ☐ The primal problem is

$$\left.\begin{array}{lll} \min Z = \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} c_{ij}\, x_{ij} & & \\ s.t. & & \\ u_i \leftrightarrow \sum\limits_{j=1}^{n} x_{ij} = a_i & i = 1, \ldots, m & \\ v_j \leftrightarrow \sum\limits_{i=1}^{m} x_{ij} = b_j & j = 1, \ldots, n & \\ \quad x_{ij} \geq 0 & & \end{array}\right\} (P)$$

# THE *STP*

- The dual problem is

$$\left.\begin{array}{lll} \max W = \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j & & \\ s.t. & & \\ x_{ij} \quad \leftrightarrow \quad u_i + v_j \leq c_{ij} & & i = 1, \ldots ,m \\ & & j = 1, \ldots ,n \\ \qquad u_{i,} v_j \text{ are unrestricted in sign} & & \end{array}\right\} \quad (D)$$

---

# THE *STP*

- The *complementary slackness conditions* for $(D)$ are

$$x_{ij}^{*}[u_i^{*} + v_j^{*} - c_{ij}] = 0 \qquad \begin{array}{l} i = 1, \ldots , m \\ j = 1, \ldots , n \end{array}$$

- Due to the equalities in $(P)$, the *complementary slackness conditions* in $(P)$ are unable to provide any additional or useful information

# THE TRANSPORTATION PROBLEM

- ☐ The *complementary slackness conditions* obtain

$$x^*_{ij} > 0 \Rightarrow u^*_i + v^*_j = c_{ij}$$

$$u^*_i + v^*_j < c_{ij} \Rightarrow x^*_{ij} = 0$$

- ☐ We make use of these *complementary slackness conditions* to develop the so–called $u - v$ *method* for solving the *standard transportation problem*

# THE $u - v$ METHOD

- ☐ The $u - v$ *method* starts with a *basic feasible solution* for the primal problem, determines the corresponding dual variables (as if the *basic feasible solution* were optimal) and uses the duals to deter–mine the *adjacent basic feasible solution*; the iteration process continues until the optimality conditions are satisfied

# THE $u - v$ METHOD

- ☐ For a *basic feasible solution*, we find the dual variable $u_i$ and $v_j$ using the *complementary slackness conditions*

$$u_i + v_j = c_{ij} \qquad \forall \text{ basic } x_{ij}$$

with $u_i$ and $v_j$ being unrestricted in sign

---

# THE $u - v$ METHOD

- ☐ We compute

$$\tilde{c}_{ij} = c_{ij} - ( u_i + v_j ) \qquad \forall \text{ nonbasic } x_{ij}$$

- ☐ This step is the analogue of computing $\underline{\tilde{c}}^T$ in the simplex tableau approach (relative cost reduction vector)
- ☐ The *complementary-slackness*-based optimality test is performed :

$$\text{if} \quad \tilde{c}_{ij} \geq 0 \quad \forall \text{nonbasic } x_{ij} \left[ x_{ij} = 0 \right]$$, then the *basic feasible solution* is *optimal*

# THE $u - v$ METHOD

- Else, we consider each nonbasic variable $x_{\bar{p}\bar{q}}$ that satisfies

$$\tilde{c}_{\bar{p}\bar{q}} = c_{\bar{p}\bar{q}} - (u_{\bar{p}} + v_{\bar{q}}) < 0$$

and determine

$$\tilde{c}_{pq} = \underbrace{\min}_{\substack{\bar{p}\bar{q} \ni x_{\bar{p}\bar{q}} \\ \text{is nonbasic} \\ \text{and } \tilde{c}_{\bar{p}\bar{q}} < 0}} \left\{ \tilde{c}_{\bar{p}\bar{q}} \right\}$$

- We, then, select $x_{pq}$ to become the next *basic variable* and repeat the process for this new *basic feasible solution*; we continue the process until the *optimality conditions* are met

# *STP* NUMERICAL EXAMPLE

- We apply the $u - v$ scheme to the example

  previously discussed

- The basic step from the dual formulation is to

  require

$$( u_i + v_j ) = c_{ij} \qquad \forall \ nonbasic \ x_{ij}$$

# *STP* NUMERICAL EXAMPLE

- We start with the *basic feasible solution* and apply the *complementary slackness conditions*

$$\begin{aligned} u_1 &+ v_4 = 1 = c_{14} \\ u_2 &+ v_4 = 4 = c_{24} \\ u_2 &+ v_3 = 5 = c_{23} \\ u_3 &+ v_2 = 6 = c_{32} \\ u_3 &+ v_1 = 7 = c_{31} \\ u_2 &+ v_1 = 10 = c_{21} \end{aligned}$$

- We have $6$ equations in $7$ unknowns and so there is an infinite number of solutions

---

# *STP* NUMERICAL EXAMPLE

- Arbitrarily, we set the variable

$$v_4 = 0$$

and solve the equations above to obtain

$$\begin{aligned} u_1 &= 1 \\ u_2 &= 4 \\ v_3 &= 1 \\ v_1 &= 6 \\ u_3 &= 1 \\ v_2 &= 5 \end{aligned}$$

# *STP* NUMERICAL EXAMPLE

- The $\tilde{c}_{ij}$ for the *nonbasic variables* are

$$
\begin{array}{llllllllll}
x_{11}: & \tilde{c}_{11} & = & c_{11} & - & (u_1+v_1) & = & 2 & - & (1+6) & = & -5 \\
x_{12}: & \tilde{c}_{12} & = & c_{12} & - & (u_1+v_2) & = & 2 & - & (1+5) & = & -4 \\
x_{13}: & \tilde{c}_{13} & = & c_{13} & - & (u_1+v_3) & = & 2 & - & (1+1) & = & 0 \\
x_{34}: & \tilde{c}_{34} & = & c_{34} & - & (u_3+v_4) & = & 8 & - & (1+0) & = & 7 \\
x_{33}: & \tilde{c}_{33} & = & c_{33} & - & (u_3+v_3) & = & 6 & - & (1+1) & = & 4
\end{array}
$$

---

# *STP* NUMERICAL EXAMPLE

- We determine

$$
\tilde{c}_{pq} = \underset{\substack{\overline{pq} \ \ni \ x_{\overline{pq}} \\ is\ nonbasic}}{min} = \tilde{c}_{11} = -5
$$

and consequently we pick the *nonbasic variable* $x_{11}$ to enter the *basis*

- We determine the maximal value of $x_{11}$ and set $x_{11} = \theta$ and make use of the tableau

# *STP* NUMERICAL EXAMPLE

| *market j* / *w/h i* | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $a_i$ |
|---|---|---|---|---|---|
| $W_1$ | $\theta$ | | | $3 - \theta$ | 3 |
| $W_2$ | $2 - \theta$ | | 4 | $1 + \theta$ | 7 |
| $W_3$ | 2 | 3 | | | 5 |
| $b_j$ | 4 | 3 | 4 | 4 | |

---

# *STP* NUMERICAL EXAMPLE

- ☐ Therefore,

$$\theta = min\ \{ 2, 3 \} = 2$$

- ☐ Consequently, $x_{21}$ becomes *0* and leaves the basis
- ☐ We obtain the *basic feasible solution*

$$x_{14} = 1,\ x_{11} = 2,\ x_{31} = 2,\ x_{32} = 3,\ x_{23} = 4,\ x_{24} = 3$$

and repeat to solve the $u - v$ problem for this adjacent *basic feasible solution*

## *STP* NUMERICAL EXAMPLE

| market $j$ / w/h $i$ | $v_1 = 2$ | $v_2 = 1$ | $v_3 = 2$ | $v_4 = 1$ | $a_i$ |
|---|---|---|---|---|---|
| $u_1 = 0$ | ② $\boxed{2}$ | $\boxed{2}$ | $\boxed{2}$ | ① $\boxed{1}$ | 3 |
| $u_2 = 3$ | $\boxed{10}$ | $\boxed{8}$ | ④ $\boxed{5}$ | ③ $\boxed{4}$ | 7 |
| $u_3 = 5$ | ② $\boxed{7}$ | ③ $\boxed{6}$ | $\boxed{6}$ | $\boxed{8}$ | 5 |
| $b_j$ | 4 | 3 | 4 | 4 | |

---

## *STP* NUMERICAL EXAMPLE

❑ The complementary slackness conditions of the nonzero valued basic variables obtain

$$u_1 + v_1 = c_{11} = 2$$

$$u_1 + v_4 = c_{14} = 1$$

$$u_2 + v_3 = c_{23} = 5$$

$$u_2 + v_4 = c_{24} = 4$$

$$u_3 + v_1 = c_{31} = 7$$

$$u_3 + v_2 = c_{32} = 6$$

## *STP* NUMERICAL EXAMPLE

☐ We set arbitrarily

$$u_1 = 0$$

and therefore

$$\begin{array}{lll} v_3 = 2 & \qquad & v_1 = 2 \\ u_3 = 5 & & u_3 = 5 \\ v_2 = 1 & & v_2 = 0 \end{array}$$

☐ We compute $\tilde{c}_{ij}$ for each nonbasic variable $x_{ij}$

---

## *STP* NUMERICAL EXAMPLE

$$\begin{array}{llllllll}
\tilde{c}_{12} = & c_{12} & - & (u_1 + v_2) & = 2 & - & (0 + 1) & = 1 \\
\tilde{c}_{13} = & c_{13} & - & (u_1 + v_3) & = 2 & - & (0 + 2) & = 0 \\
\tilde{c}_{21} = & c_{21} & - & (u_2 + v_1) & = 10 & - & (3 + 2) & = 5 \\
\tilde{c}_{22} = & c_{22} & - & (u_2 + v_2) & = 8 & - & (3 + 1) & = 4 \\
\tilde{c}_{33} = & c_{33} & - & (u_3 + v_3) & = 6 & - & (5 + 2) & = -1 \\
\tilde{c}_{34} = & c_{34} & - & (u_3 + v_4) & = 8 & - & (5 + 1) & = 2
\end{array}$$

only possible improvement

☐ We introduce $x_{33}$ as a *basic variable* and determine its *nonnegative value* $\theta$ from the tableau

## *STP* NUMERICAL EXAMPLE

| market *j* / w/h *i* | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $a_i$ |
|---|---|---|---|---|---|
| $W_1$ | $2 + \theta$ | | | $1 - \theta$ | 3 |
| $W_2$ | | | $4 - \theta$ | $3 + \theta$ | 7 |
| $W_3$ | $2 - \theta$ | 3 | $\theta$ | | 5 |
| $b_j$ | 4 | 3 | 4 | 4 | |

---

## *STP* NUMERICAL EXAMPLE

- ❑ The limiting value of $\theta$ is

$$\theta = min \{ 2, 4, 1 \} = 1$$

- ❑ Consequently, $x_{14}$ leaves the basis and $x_{33}$ enters the basis with the value 1

- ❑ We obtain the adjacent basic feasible solution in

## *STP* NUMERICAL EXAMPLE

| market *j*<br>w/h *i* | $v_1 = 2$ | $v_2 = 1$ | $v_3 = 1$ | $v_4 = 0$ | $a_i$ |
|---|---|---|---|---|---|
| $u_1 = 0$ | ③ $\boxed{2}$ | $\boxed{2}$ | $\boxed{2}$ | $\boxed{1}$ | 3 |
| $u_2 = 4$ | $\boxed{10}$ | $\boxed{8}$ | ③ $\boxed{5}$ | ④ $\boxed{4}$ | 7 |
| $u_3 = 5$ | ① $\boxed{7}$ | ③ $\boxed{6}$ | ① $\boxed{6}$ | $\boxed{8}$ | 5 |
| $b_j$ | 4 | 3 | 4 | 4 | |

---

## *STP* NUMERICAL EXAMPLE

- ☐ We evaluate $\tilde{c}_{ij}$ for each nonbasic variable; $\tilde{c}_{ij} \geq 0$ and so we have *an optimal solution* with

  *shipping* 3 *from* $W_1$ *to* $M_1$ *with costs* 6

  *shipping* 1 *from* $W_3$ *to* $M_1$ *with costs* 7

  *shipping* 3 *from* $W_3$ *to* $M_2$ *with costs* 18

  *shipping* 1 *from* $W_3$ *to* $M_3$ *with costs* 6

  *shipping* 3 *from* $W_2$ *to* $M_3$ *with costs* 15

  *shipping* 4 *from* $W_2$ *to* $M_4$ *with costs* 16

  and resulting in the *least total costs* of 68

# ELECTRICITY DISTRIBUTION EXAMPLE

☐ We consider an electric utility system in which 3 power plants are used to supply the electricity demand of 4 cities

☐ The supplies available from the 3 plants are given

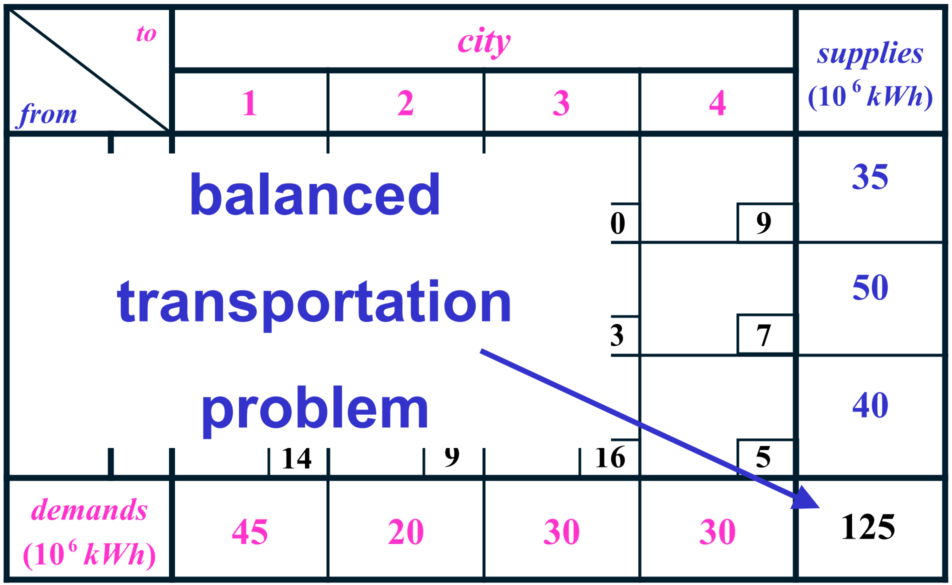
☐ The demands of the 4 cities are specified

☐ The costs of supply per $10^6\ kWh$ are given

# ELECTRICITY COSTS

| *to* *from* | | *city* | | | | *supplies* ($10^6\ kWh$) |
|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | |
| *plant* | 1 | 8 | 6 | 10 | 9 | 35 |
| | 2 | 9 | 12 | 13 | 7 | 50 |
| | 3 | 14 | 9 | 16 | 5 | 40 |
| *demands* ($10^6\ kWh$) | | 45 | 20 | 30 | 30 | 125 |

# ELECTRICITY COSTS



| *to* / *from* | *city* | | | | *supplies* ($(10^6\ kWh)$) |
| --- | --- | --- | --- | --- | --- |
| | 1 | 2 | 3 | 4 | |
| | | | 0 | 9 | 35 |
| | | | 3 | 7 | 50 |
| | 14 | 9 | 16 | 5 | 40 |
| *demands* ($(10^6\ kWh)$) | 45 | 20 | 30 | 30 | 125 |

**balanced transportation problem**

---

# ELECTRICITY ALLOCATION EXAMPLE

- ☐ We note that

$$\sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j$$

  and so we have a balanced transportation problem

- ☐ We make use of the *LCRP* to construct a basic feasible solution

# ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

| *from* \ *to* | | *city* 1 | 2 | 3 | 4 | *supplies* ($10^6$ *kWh*) |
|---|---|---|---|---|---|---|
| *plant* | 1 | 8 | 6 | 10 | 0<br>9 | 35 |
| | 2 | 9 | 12 | 13 | 0<br>7 | 50 |
| | 3 | 14 | 9 | 16 | 30<br>5 | 10 |
| *demands* ($10^6$ *kWh*) | | 45 | 20 | 30 | 30 | 125 |

---

# ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

❑ And we set

$$x_{34} = 30$$

$$x_{14} = 0$$

$$x_{24} = 0$$

❑ We compute the remaining supply at plant 3 and remove column corresponding to city 4 from further consideration

❑ We continue with the reduced system

# ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

| to<br>from | | city | | | supplies<br>$(10^6\,kWh)$ |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | |
| plant | 1 | 8 | 20<br>6 | 10 | 15 |
| | 2 | 9 | 0<br>12 | 13 | 50 |
| | 3 | 14 | 0<br>9 | 16 | 10 |
| demands<br>$(10^6\,kWh)$ | | 45 | 20 | 30 | |

---

# ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

and so we set

$$x_{12} = 20$$

$$x_{22} = 0$$

$$x_{32} = 0$$

- ☐ We recompute the supply remaining at plant 1 and remove column corresponding to city 2
- ☐ The new reduced system obtains

# ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

| *to*<br>*from* | | *city* | | *supplies*<br>($10^6$ *kWh*) |
|---|---|---|---|---|
| | | 1 | 3 | |
| *plant* | 1 | 15<br>8 | 0<br>10 | 15 |
| | 2 | 9 | 13 | 50 |
| | 3 | 14 | 16 | 10 |
| *demands*<br>($10^6$ *kWh*) | | 30 | 30 | |

---

# ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

and therefore we set

$$x_{11} = 15$$

$$x_{13} = 0$$

and remove the row corresponding to plant 1 from further consideration since its supply is exhausted

☐ The operation is repeated on the reduced system

# ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

| *from* \ *to* | | *city* 1 | *city* 3 | *supplies* ($10^6$ *kWh*) |
|---|---|---|---|---|
| *plant* | 2 | 30<br>9 | 13 | 20 |
| | 3 | *0*<br>14 | 16 | 10 |
| *demands* ($10^6$ *kWh*) | | 30 | 30 | |

---

# ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

and therefore we set

$$x_{21} = 30$$

$$x_{31} = 0$$

and remove the column corresponding to city 1 from further consideration

- We are finally left with

# ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

| to / from | | city | supplies ($10^6$ *kWh*) |
|---|---|---|---|
| | | 3 | |
| *plant* | 2 | 20<br>13 | 20 |
| | 3 | 10<br>16 | 10 |
| *demands* ($10^6$ *kWh*) | | 30 | |

---

# ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

which allows us to set

$$x_{23} = 20$$

$$x_{33} = 10$$

- The basic feasible solution has the costs

$$Z = 30 \cdot 5 + 20 \cdot 6 + 15 \cdot 8 + 30 \cdot 9 + 20 \cdot 13 + 10 \cdot 16 = 1{,}080$$

- We improve this solution by using the $u - v$ *scheme*
- The first tableau corresponding to the initial basic feasible solution is:

# ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

| *to*<br>*from* | | *city* | | | | *supplies* $(10^6\,kWh)$ |
|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | |
| *plant* | 1 | ⑮ $\boxed{8}$ | ⑳ $\boxed{6}$ | | | 35 |
| | 2 | ㉚ $\boxed{9}$ | | ⑳ $\boxed{13}$ | | 50 |
| | 3 | | | ⑩ $\boxed{16}$ | ㉚ $\boxed{5}$ | 40 |
| *demands* $(10^6\,kWh)$ | | 45 | 20 | 30 | 30 | |

---

# *STP* NUMERICAL EXAMPLE

- ❑ We compute, the possible improvements at each nonbasic variable:

$$
\begin{aligned}
\tilde{c}_{31} &= c_{31} - (u_3 + v_1) = 14 - (4+8) = 2 \\
\tilde{c}_{22} &= c_{22} - (u_2 + v_2) = 12 - (1+6) = 5 \\
\tilde{c}_{32} &= c_{32} - (u_3 + v_2) = 9 - (4+6) = -1 \\
\tilde{c}_{13} &= c_{13} - (u_1 + v_3) = 10 - (0+12) = -2 \\
\tilde{c}_{14} &= c_{14} - (u_1 + v_4) = 9 - (0+1) = 8 \\
\tilde{c}_{24} &= c_{24} - (u_2 + v_4) = 7 - (1+1) = 5
\end{aligned}
$$

*improvement possible*

*better improvement*

## *STP* NUMERICAL EXAMPLE

- ☐ We bring $x_{13}$ into the basis and determine the value of $\theta$ using the tableau structure
- ☐ From the tableau we conclude that

$$\theta = min\ \{ 15, 20 \} = 15$$

and therefore $x_{11}$ leaves the basis to determine the *adjacent basic feasible solution* given in the table

---

## *STP* NUMERICAL EXAMPLE

| *cities*<br>*plants* | 1 | 2 | 3 | 4 | $a_i$ |
|---|---|---|---|---|---|
| 1 | $15 - \theta$ | 20 | $\theta$ | | 35 |
| 2 | $30 + \theta$ | | $20 - \theta$ | | 50 |
| 3 | | | 10 | 30 | 40 |
| $b_j$ | 45 | 20 | 30 | 30 | |

# *STP* NUMERICAL EXAMPLE

- The adjacent basic feasible solution is

$$x_{21} = 45, \quad x_{12} = 20, \quad x_{13} = 15, \quad x_{23} = 5, \quad x_{33} = 10, \quad x_{34} = 30$$

  and the new value of $Z$ is

$$Z \;=\; 20 \cdot 6 + 15 \cdot 10 + 45 \cdot 9 + 5 \cdot 13 \;+\; 10 \cdot 16 \;+\; 30 \cdot 5$$

$$=\; 1050 \;<\; 1080$$

- We again pursue a $u - v$ improvement strategy by starting with the tableau

---

# *STP* NUMERICAL EXAMPLE

| *cities*<br>*plants* | $v_1 = 6$ | $v_2 = 6$ | $v_3 = 10$ | $v_4 = -1$ | *supplies* |
|---|---|---|---|---|---|
| $u_1 = 0$ |  | ⑳<br>6 | ⑮<br>10 |  | 35 |
| $u_2 = 3$ | ㊺<br>9 |  | ⑤<br>13 |  | 50 |
| $u_3 = 6$ |  |  | ⑩<br>16 | ㉚<br>5 | 40 |
| *demands* | 45 | 20 | 30 | 30 |  |

# STANDARD TRANSPORTATION EXAMPLE

- ☐ The complementary slackness conditions obtain the possible improvements

$$
\begin{array}{lllllllll}
\tilde{c}_{11} & = & c_{11} & - & (u_1+v_1) & = & 8 & - & (0+6) & = & 2 \\
\tilde{c}_{31} & = & c_{31} & - & (u_3+v_1) & = & 14 & - & (6+6) & = & 2 \\
\tilde{c}_{22} & = & c_{22} & - & (u_2+v_2) & = & 12 & - & (3+6) & = & 3 \\
\tilde{c}_{32} & = & c_{32} & - & (u_3+v_2) & = & 9 & - & (6+6) & = & \boxed{-3} \\
\tilde{c}_{14} & = & c_{14} & - & (u_1+v_4) & = & 9 & - & (0-1) & = & 10 \\
\tilde{c}_{24} & = & c_{24} & - & (u_2+v_4) & = & 7 & - & (3-1) & = & 5
\end{array}
$$

only possible improvement

- ☐ We bring $x_{32}$ into the basis and with its value $\theta$ determined from

# *STP* NUMERICAL EXAMPLE

| plants / cities | 1 | 2 | 3 | 4 | $a_i$ |
|---|---|---|---|---|---|
| 1 | | 20 $-\theta$ | 15 $+\theta$ | | 35 |
| 2 | 45 | | 5 | | 50 |
| 3 | | $\theta$ | 10 $-\theta$ | 30 | 40 |
| $b_j$ | 45 | 20 | 30 | 30 | |

# *STP* NUMERICAL EXAMPLE

and so

$$\theta = min \{ 10, 20 \} = 10$$

- The adjacent basic feasible solution is, then,

$$x_{21} = 45 \qquad x_{12} = 10 \qquad x_{32} = 10$$

$$x_{13} = 25 \qquad x_{23} = 5 \qquad x_{34} = 30$$

and the value of $Z$ becomes

$$Z = 45 \cdot 9 + 10 \cdot 6 + 10 \cdot 9 + 25 \cdot 10 + 5 \cdot 13 + 30 \cdot 5 = 1{,}02$$

- You need to prove, using *complementary slackness conditions*, that this is the true optimum

# NONSTANDARD TRANSPORTATION PROBLEM

- The nonstandard transportation problem arises when supply and demand are unbalanced: either supply exceeds demand or vice versa
- We solve by transforming the nonstandard problem into a standard one
- The approach is to create a *fictitious* entity – a market to absorb the surplus supply or a warehouse for the supply deficit – and solve the problem with the fictitious entity as a balanced transportation problem

# NONSTANDARD TRANSPORTATION PROBLEM

- For the case

$$\underbrace{\sum_{i=1}^{m} a_i}_{\text{supply}} \quad > \quad \underbrace{\sum_{i=1}^{n} b_j}_{\text{demand}}$$

we create the fictitious market $M_{n+1}$ to absorb all the excess supply $\left( \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j \right)$; we set $c_{i,n+1} = 0$, $\forall \; i = 1, 2, \ldots, m$ since $M_{n+1}$ is fictitious

- The problem is then in standard form with $j = 1, 2, \ldots, n, n+1$, for the augmented number of markets

---

# NONSTANDARD TRANSPORTATION PROBLEM

- For the case

$$\underbrace{\sum_{j=1}^{n} b_j}_{\text{demand}} \quad > \quad \underbrace{\sum_{i=1}^{m} a_i}_{\text{supply}}$$

the problem is *not* , in effect, *feasible* since all the demands cannot be met and therefore the least–cost shipping schedule is that which will supply as much as possible of the demands of the markets at the lowest cost

# NONSTANDARD TRANSPORTATION PROBLEM

- ☐ For the excess demand case, we introduce the fictitious warehouse $W_{m+1}$ to supply the shortage $\left[\sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i\right]$ and we set $c_{m+1,j} = 0, j = 1, 2, \ldots, n$
- ☐ The problem is in standard form with $i = 1, \ldots, m + 1$ (number of warehouses augmented by 1)

# NONSTANDARD TRANSPORTATION PROBLEM

- ☐ Note that the variable $x_{m+1,j}$ is the *shortage* at market $j$ and is the shortfall in the demand $b_j$ experienced by each market $M_j$ due to inadequate supplies at warehouses $i = 1, 2, \ldots, m$
- ☐ At each market $j$ , $x_{m+1,j}$ provides the measure of the *infeasibility* of the problem

## EXAMPLE: CANNING OPERATIONS SCHEDULING

- ❑ **This problem is concerned with the scheduling the purchases of 2 plants – *A* and *B* – of the raw supplies from 3 growers with given availability / price**

| *grower* | *availability* (*ton*) | *price* ( *$ / ton* ) |
|---|---|---|
| ***Smith*** | **200** | **10** |
| ***Jones*** | **300** | **9** |
| ***Richard*** | **400** | **8** |

## EXAMPLE: CANNING OPERATIONS SCHEDULING

- ❑ **The shipping costs in *$/ton* are given by**

| *to*<br>*from* | *plant* | |
|---|---|---|
| | ***A*** | ***B*** |
| ***Smith*** | **2** | **2.5** |
| ***Jones*** | **1** | **1.5** |
| ***Richard*** | **5** | **3** |

# EXAMPLE: CANNING OPERATIONS SCHEDULING

- The plants' capacity limits and labor costs are

| *plant* | *A* | *B* |
|---|---|---|
| *capacity* ( *ton* ) | 450 | 550 |
| *labor costs* ( *$ / ton* ) | 25 | 20 |

---

# EXAMPLE: CANNING OPERATIONS SCHEDULING

- The competitive selling price for canned goods is 50 *$ / ton* and the company can sell all it produces
- The problem is to determine the purchase schedule that produces the *maximum* profits
- Note that this is an unbalanced problem since

$$\begin{aligned} supply &= 200 + 300 + 400 = 900 \ tons \\ demand &= 450 + 550 = 1000 \ tons > 900 \ tons \end{aligned}$$

- The decision variables are the amounts bought from each grower and shipped to each plant

# EXAMPLE: CANNING OPERATIONS SCHEDULING

- ☐ The objective is formulated as

$$\max Z = \left[\underbrace{50-25-10-2}_{13}\right] x_{SA} + \left[\underbrace{50-25-9-1}_{15}\right] x_{JA}$$

$$+ \left[\underbrace{50-25-8-5}_{12}\right] x_{RA} + \left[\underbrace{50-20-10-2.5}_{17.5}\right] x_{SB}$$

$$+ \left[\underbrace{50-20-9-1.5}_{19.5}\right] x_{JB} + \left[\underbrace{50-20-8-3}_{19}\right] x_{RB}$$

---

# EXAMPLE: CANNING OPERATIONS SCHEDULING

- ☐ The supply constraints are

$$x_{SA} + x_{SB} \leq 200$$

$$x_{JA} + x_{JB} \leq 300$$

$$x_{RA} + x_{RB} \leq 400$$

- ☐ The demand constraints are

$$x_{SA} + x_{JA} + x_{RA} \leq 450$$

$$x_{SB} + x_{JA} + x_{RB} \leq 550$$

# EXAMPLE: CANNING OPERATIONS SCHEDULING

- Clearly, all decision variables are nonnegative
- The unbalanced nature of the problem requires the introduction of a *fictitious* grower *F*, who is able to supply 100 *tons* of the supply shortage; the addition of *F* allows the *nonstandard* problem to be restated as a *standard transportation problem*
- We set up the *STP* tableau

---

# EXAMPLE: CANNING OPERATIONS SCHEDULING

| *plant j* / *grower i* | *A* | *B* | *supply* |
|---|---|---|---|
| ***S*** | 13 | 17.5 | **200** |
| ***J*** | 15 | 19.5 | **300** |
| ***R*** | 12 | 19 | **400** |
| ***F*** | *0* | *0* | **100** |
| *demand* | 450 | 550 | **1,000** |

## EXAMPLE: CANNING OPERATIONS SCHEDULING

- **In this problem, the objective is a *maximization* rather than a *minimization***
- **We therefore recast the “mechanics” of the $u - v$ scheme for the *maximization* problem**
- **As a homework exercise, show that the duality complementary slackness conditions allow us to change the $u - v$ algorithm in the following way:**

## EXAMPLE: CANNING OPERATIONS SCHEDULING

- **the selection of the nonbasic variable $x_{ij}$ to enter the basis is from those $x_{ij}$ whose corresponding**

  $$c_{ij} > u_i + v_j$$

  **and we focus on and evaluate all $\tilde{c}_{ij} > 0$ for which $x_{ij}$ is a candidate to enter the basis**
- **we pick $x_{pq}$ corresponding to**

  $$\tilde{c}_{pq} = \underbrace{max}_{\substack{\bar{p}\,\bar{q} \ \ni \ x_{\bar{p}\bar{q}} \\ is\ nonbasic \\ and\ \tilde{c}_{\bar{p}\bar{q}} > 0}} \left\{ \tilde{c}_{\bar{p}\bar{q}} \right\}$$

# EXAMPLE SOLUTION

| plant j / grower i | A | B | supply |
|---|---|---|---|
| S | 200<br>13 | 0<br>17.5 | 200 |
| J | 250<br>15 | 50<br>19.5 | 300 |
| R | 0<br>12 | 400<br>19 | 400 |
| F | 0<br>0 | 100<br>0 | 100 |
| demand | 450 | 550 | |

---

# EXAMPLE SOLUTION

- We construct the $u - v$ relations for this solution

$$u_1 + v_1 = 13 \qquad u_2 + v_2 = 19.5$$

$$u_2 + v_1 = 15 \qquad u_3 + v_2 = 19$$

$$u_4 + v_2 = 0$$

- We arbitrarily set $u_1 = 0$ and compute

$$v_1 = 13, \; u_2 = 2, \; v_2 = 17.5, \; u_3 = 1.5, \; u_4 = -17.5$$

# EXAMPLE SOLUTION

- We evaluate the $\tilde{c}_{ij}$ corresponding to the nonbasic variables

$$\tilde{c}_{31} = c_{31} - (u_3 + v_1) = 12 \quad - \quad (1.5 + 13) \quad = -\,2.5$$

$$\tilde{c}_{41} = c_{41} - (u_4 + v_1) = \; 0 \quad - (-17.5 + 13) \quad = \quad 4.5$$

$$\tilde{c}_{12} = c_{12} - (u_1 + v_2) = 17.5 - \quad (0 + 17.5) \quad = \quad 0$$

single possible improvement

- Thus, $x_{41}$ enters the basis and we determine $\theta$

---

# EXAMPLE SOLUTION

| plant j<br>grower i | A | B | supply |
|---|---|---|---|
| S | 200<br>13 | | 200 |
| J | 250 – $\theta$<br>15 | 50 + $\theta$<br>19.5 | 300 |
| R | | 400<br>19 | 400 |
| F | $\theta$<br>0 | 100 – $\theta$<br>0 | 100 |
| demand | 450 | 550 | |

# EXAMPLE SOLUTION

- ☐ It follows that

$$\theta \;=\; min\;\{\,250,\,100\,\}\;=\;100$$

and so the adjacent basic feasible solution is

$$x_{11} = 200,\;\; x_{21} = 150,\;\; x_{41} = 100,\;\; x_{22} = 150,\;\; x_{32} = 400$$

- ☐ We repeat the $u - v$ procedure with the new *basic variables* and solve

---

# EXAMPLE SOLUTION

$$u_1 + v_1 \;=\; 13 \qquad\qquad u_2 + v_2 \;=\; 19.5$$

$$u_2 + v_1 \;=\; 15 \qquad\qquad u_3 + v_2 \;=\; 19$$

$$u_4 + v_1 \;=\; 0$$

- ☐ We solve by arbitrarily setting $u_1 = 0$ and obtain

$$v_1 \;=\; 13\,,\;\; u_2 \;=\; 2\,,\;\; v_2 \;=\; 17.5\,,\;\; u_3 \;=\; 1.5\,,\;\; u_4 \;=\; -13$$

# EXAMPLE SOLUTION

- We compute the $\tilde{c}_{ij}$ for the nonbasic variables

$$\tilde{c}_{12} = 17.5 - (0 + 17.5) = 0$$

$$\tilde{c}_{31} = 12 - (1.5 + 13) = -2.5$$

$$\tilde{c}_{42} = 0 - (-13 + 17.5) = -4.5$$

---

# EXAMPLE SOLUTION

- Since each $\tilde{c}_{ij}$ is $\leq 0$ , no improvement in the maximization is possible and so the maximum profits are

$$Z = (200)13 + (150)15 + (100)0 + (150)19.5 + (400)19$$

$$= 15{,}375 \ \$$$

# SCHEDULING PROBLEM AS A STANDARD TRANSPORTATION PROBLEM

- The problem is concerned with the weekly production scheduling over a 4 – week period
  - production costs for each item

| | |
|---|---|
| *first two weeks* | **$ 10** |
| *last two weeks* | **$ 15** |

  - demands that need to be met are

| *week* | **1** | **2** | **3** | **4** |
|---|---|---|---|---|
| *demand* | **300** | **700** | **900** | **800** |

---

# SCHEDULING PROBLEM AS A STANDARD TRANSPORTATION PROBLEM

  - weekly plant capacity is 700
  - overtime is possible for weeks 2 and 3 with
    - – the production of additional 200 *units*
    - – additional cost per unit of *$* 5
  - *$* 3 for weekly storage of unsold production
  - the objective is to *minimize* the *total costs* for the 4–week schedule
- The decision variables are

$x_{ij}$ = *production in week i for use in week j market*

# SCHEDULING PROBLEM AS A STANDARD TRANSPORTATION PROBLEM

| *demand wk.* / *production wk.* | | 1 | 2 | 3 | 4 | *F* | *supply* |
|---|---|---|---|---|---|---|---|
| 1 | | 10 | 13 | 16 | 19 | *0* | 700 |
| 2 | *normal* | *M* | 10 | 13 | 16 | *0* | 700 |
| | *o/t* | *M* | 15 | 18 | [illegible] | *0* | 200 |
| 3 | *normal* | *M* | *M* | 15 | 18 | *0* | 700 |
| | *o/t* | *M* | [illegible] | [illegible] | [illegible] | *0* | 200 |
| 4 | | *M* | *M* | *M* | 15 | *0* | 700 |
| *demand* | | 300 | 700 | 900 | 800 | 500 | |

*M* is a very large number

3,200

2,700

3,200 – 2,700

---

# ASSIGNMENT PROBLEM

- ☐ We are given

$n$ machines $M_1, M_2, \ldots, M_n \leftrightarrow i$

$n$ jobs $J_1, J_2, \ldots, J_n \leftrightarrow j$

$c_{ij} =$ cost of doing job $j$ on machine $i$

$c_{ij} = M$ if job $j$ cannot be done on machine $i$

each machine can only do one job and we wish to determine the optimal match, *i.e.*, the assignment with the lowest total costs of doing each job $j$ on the $n$ available machines

## ASSIGNMENT PROBLEM

- The brute force approach is simply enumeration: consider $n = 10$ and there are $3,628,800$ possible choices!
- We can, however, introduce *categorical* decision variables

$$x_{ij} = \begin{cases} 1 & \text{job } j \text{ is assigned to machine } i \\ 0 & \text{otherwise} \end{cases}$$

## ASSIGNMENT PROBLEM

and the problem constraints can be stated as

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \forall i \quad \text{each machine does exactly } 1 \text{ job}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \forall j \quad \text{each job is assigned to } 1 \text{ machine}$$

- The objective, then, is

$$min\ Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

# ASSIGNMENT PROBLEM

- ☐ This assignment problem is an *STP* with

$$a_i = 1 \qquad \forall i$$

$$b_j = 1 \qquad \forall j$$

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} b_j$$

# NONSTANDARD ASSIGNMENT PROBLEM

- ☐ Suppose we have $m$ machines and $n$ jobs with $m \neq n$
- ☐ We may convert this into an equivalent *standard assignment problem* with *equal* number of machines and jobs
- ☐ The conversion requires the introduction of either *fictitious* jobs or *fictitious* machines

# NONSTANDARD ASSIGNMENT PROBLEM

❑ In the case $m > n$ :

we create ( $m - n$ ) fictitious jobs and we have $m$ machines and $n + m - n = m$ jobs; we assign the machinery costs for the fictitious jobs to be *0* : note that the objective function *remains unchanged* since a fictitious job assigned to a machine is, in effect, a machine that remains *idle*

# NONSTANDARD ASSIGNMENT PROBLEM

❑ For the case $n > m$ :

we create ( $n - m$ ) *fictitious* machines with machine costs of *0* and the solution obtained has the ( $n - m$ ) jobs that cannot be done due to lack of machines

# NONSTANDARD ASSIGNMENT PROBLEM

- ☐ **In principle, any assignment problem may be solved using the transportation problem technique; in practice, this approach is not practical since every basic feasible solution is** ***degenerate***
- ☐ **We note that in the** ***standard assignment problem*** **for** $m$ **machines with** $m = n$ **, there are exactly** $m$ $x_{ij}$ **that are** $1$ **(*nonzero*) but** ***every basic feasible solution*** **of the transportation problem has** $(2m - 1)$ **basic variables of which** $(m - 1)$ **have the value zero**