ECE 307- Techniques for Engineering Decisions

Lecture 4. Duality Concepts in Linear Programming

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DUALITY

- **Definition: A *LP* is in *symmetric form* if all the variables are restricted to be *nonnegative* and all the constraints are inequalities of the type:**

objective type	*corresponding inequality type*
max	$\leq$
min	$\geq$

DUALITY DEFINITIONS

- We first define the *primal* and *dual* problems

$$\left.\begin{array}{ll} \max & Z = \underline{c}^T \underline{x} \\ s.t. & \\ & \underline{A}\,\underline{x} \leq \underline{b} \\ & \underline{x} \geq \underline{0} \end{array}\right\} (P)$$

$$\left.\begin{array}{ll} \min & W = \underline{b}^T \underline{y} \\ s.t. & \\ & \underline{A}^T \underline{y} \geq \underline{c} \\ & \underline{y} \geq \underline{0} \end{array}\right\} (D)$$

DUALITY DEFINITIONS

- The problems (P) and (D) are called the *symmetric dual LP* problems; we restate them as

$$
\left.
\begin{array}{l}
\max Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \\
s.t. \\
a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \leq b_1 \\
a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \leq b_2 \\
\vdots \\
a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \leq b_m \\
x_1 \geq 0, \quad x_2 \geq 0, \quad \ldots, \quad x_n \geq 0
\end{array}
\right\} (P)
$$

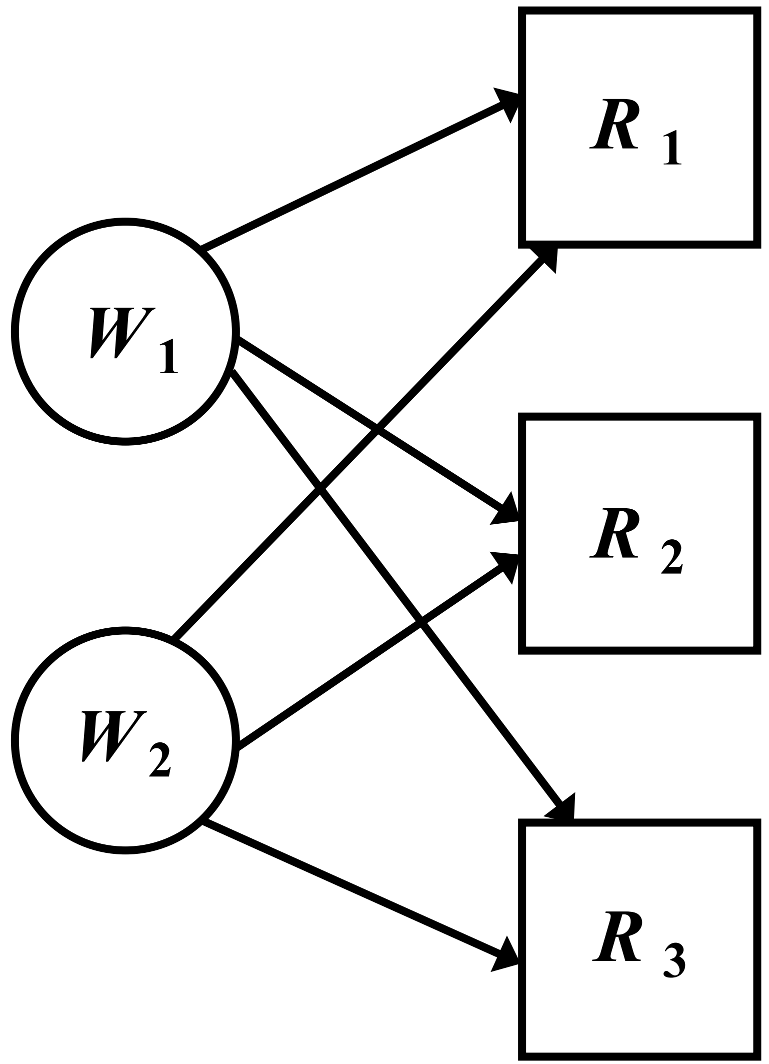
DUALITY DEFINITIONS

$$\left.\begin{array}{l}
\min W = b_1 y_1 + b_2 y_2 + \dots + b_m y_m \\
s.t. \\
a_{11} y_1 + a_{21} y_2 + \dots + a_{m1} y_m \geq c_1 \\
a_{12} y_1 + a_{22} y_2 + \dots + a_{m2} y_m \geq c_2 \\
\vdots \\
a_{1n} y_1 + a_{2n} y_2 + \dots + a_{mn} y_m \geq c_n \\
y_1 \geq 0, \quad y_2 \geq 0, \quad \dots, \quad y_m \geq 0
\end{array}\right\} (D)$$

EXAMPLE 1: MANUFACTURER TRANSPORTATION PROBLEM

shipment cost coefficients

warehouses	*retail stores*		
	R_1	R_2	R_3
W_1	2	4	3
W_2	5	3	4



EXAMPLE 1: MANUFACTURER TRANSPORTATION PROBLEM

- We are given that the *supplies* stored in warehouses W_1 and W_2 satisfy

$$supply\ at\ W_1 \leq 300$$

$$supply\ at\ W_2 \leq 600$$

- We are also given the *demands needed* to be met at the retail stores R_1, R_2, and R_3 :

$$\textbf{\textit{demand at}}\ R_1 \geq 200$$

$$\textbf{\textit{demand at}}\ R_2 \geq 300$$

$$\textbf{\textit{demand at}}\ R_3 \geq 400$$

EXAMPLE 1: MANUFACTURER TRANSPORTATION PROBLEM

- The problem is to determine the *least-cost* shipping schedule

- We define the decision variable

$$x_{ij} = \textit{quantity shipped from } W_i \textit{ to } R_j \quad i = 1,2, \;\; j = 1,2,3$$

- The shipping costs may be viewed as

$$c_{ij} = \textit{element } i, j \textit{ of the transportation cost matrix}$$

FORMULATION STATEMENT

$$\min Z = \sum_{i=1}^{2}\sum_{j=1}^{3} c_{ij}\, x_{ij} = 2x_{11} + 4x_{12} + 3x_{13} + 5x_{21} + 3x_{22} + 4x_{23}$$

s.t.

$$\begin{array}{lllllll}
x_{11} + x_{12} + x_{13} & & & & & \leq & 300 \\
 & x_{21} + x_{22} + x_{23} & & & & \leq & 600 \\
x_{11} & + x_{21} & & & & \geq & 200 \\
x_{12} & + x_{22} & & & & \geq & 300 \\
x_{13} & + x_{23} & & & & \geq & 400
\end{array}$$

$$x_{ij} \geq 0 \quad i = 1,2, \;\; j = 1,2,3$$

DUAL PROBLEM SETUP USING SYMMETRIC FORM

$$\min Z = \sum_{i=1}^{2}\sum_{j=1}^{3} c_{ij}\, x_{ij}$$

s.t.

$$\begin{array}{lcrrrrrrl}
y_1 & \leftrightarrow & -x_{11} & -x_{12} & -x_{13} & & & & \geq -300 \\
y_2 & \leftrightarrow & & & & -x_{21} & -x_{22} & -x_{23} & \geq -600 \\
y_3 & \leftrightarrow & x_{11} & & & +x_{21} & & & \geq 200 \\
y_4 & \leftrightarrow & & x_{12} & & & +x_{22} & & \geq 300 \\
y_5 & \leftrightarrow & & & x_{13} & & & +x_{23} & \geq 400
\end{array}$$

$$x_{ij} \geq 0 \quad i = 1,2 \quad j = 1,2,3$$

DUAL PROBLEM SETUP

$$\max W = -300y_1 - 600y_2 + 200y_3 + 300y_4 + 400y_5$$

s.t.

$$
\begin{array}{rrrrrcl}
-y_1 & & +y_3 & & & \leq & c_{11} = 2 \\
-y_1 & & & +y_4 & & \leq & c_{12} = 4 \\
-y_1 & & & & +y_5 & \leq & c_{13} = 3 \\
 & -y_2 & +y_3 & & & \leq & c_{21} = 5 \\
 & -y_2 & & +y_4 & & \leq & c_{22} = 3 \\
 & -y_2 & & & +y_5 & \leq & c_{23} = 4
\end{array}
$$

$$y_i \geq 0 \quad i = 1, 2, \ldots, 5$$

THE *DUAL PROBLEM* INTERPRETATION

- The moving company proposes to the manufacturer to:

  buy all the 300 units at W_1 at y_1 / *unit*

  buy all the 600 units at W_2 at y_2 / *unit*

  sell all the 200 units at R_1 at y_3 / *unit*

  sell all the 300 units at R_2 at y_4 / *unit*

  sell all the 400 units at R_3 at y_5 / *unit*

- To convince the manufacturer to get the business, the mover ensures that the delivery fees cannot exceed the transportation costs the manufacturer would incur (the dual constraints)

THE *DUAL PROBLEM* INTERPRETATION

$$
\begin{array}{rrrrrrlcl}
-y_1 & & +y_3 & & & \le & c_{11} & = & 2 \\
-y_1 & & & +y_4 & & \le & c_{12} & = & 4 \\
-y_1 & & & & +y_5 & \le & c_{13} & = & 3 \\
& -y_2 & +y_3 & & & \le & c_{21} & = & 5 \\
& -y_2 & & +y_4 & & \le & c_{22} & = & 3 \\
& -y_2 & & & +y_5 & \le & c_{23} & = & 4
\end{array}
$$

- **The mover wishes to maximize profits, i.e.,** ***revenues – costs ⇒ dual cost objective function***

$$
\max W = -300\,y_1 - 600\,y_2 + 200\,y_3 + 300\,y_4 + 400\,y_5
$$

EXAMPLE 2: FURNITURE PRODUCTS

- **Resource requirements**

item	*sales price ($)*
desks	**60**
tables	**30**
chairs	**20**

- *requirements*
  - *lumber board*
  - *labor*
    - *finishing*
    - *carpentry*

EXAMPLE 2: FURNITURE PRODUCTS

- The Dakota Furniture Company manufacturing:

resource	*desk*	*table*	*chair*	*available*
lumber board (ft)	8	6	1	48
finishing (h)	4	2	1.5	20
carpentry (h)	2	1.5	0.5	8

- We assume that the demand for desks, tables and chairs is *unlimited* and the two required resources – lumber and labor – are already purchased
- The decision problem is to **maximize** *total revenues*

PRIMAL AND DUAL PROBLEM FORMULATION

- We define decision variables

$$x_1 = \textit{number of desks produced}$$

$$x_2 = \textit{number of tables produced}$$

$$x_3 = \textit{number of chairs produced}$$

- The Dakota problem is

$$\max \quad Z = 60x_1 + 30x_2 + 20x_3$$

$$s.t.$$

$$\begin{array}{llll} y_1 \leftrightarrow & 8x_1 + 6x_2 + x_3 \leq 48 & & \textit{lumber} \\ y_2 \leftrightarrow & 4x_1 + 2x_2 + 1.5x_3 \leq 20 & & \textit{finishing} \\ y_3 \leftrightarrow & 2x_1 + 1.5x_2 + 0.5x_3 \leq 8 & & \textit{carpentry} \end{array}$$

$$x_1, x_2, x_3 \geq 0$$

PRIMAL AND DUAL PROBLEM FORMULATION

- The dual problem is

$$\min \quad W = 48 y_1 + 20 y_2 + 8 y_3$$

$$s.t.$$

$$\begin{array}{lllllll} 8 y_1 & + & 4 y_2 & + & 2 y_3 & \geq 60 & \textit{desk} \\ 6 y_1 & + & 2 y_2 & + & 1.5 y_3 & \geq 30 & \textit{table} \\ y_1 & + & 1.5 y_2 & + & 0.5 y_3 & \geq 20 & \textit{chair} \end{array}$$

$$y_1 , y_2 , y_3 \geq 0$$

PRIMAL AND DUAL PROBLEM FORMULATION

$$\max \quad Z = 60x_1 + 30x_2 + 20x_3$$

$$\begin{array}{llllll}
y_1 \leftrightarrow & 8x_1 + & 6x_2 + & x_3 & \leq 48 & \textit{lumber} \\
y_2 \leftrightarrow & 4x_1 + & 2x_2 + & 1.5x_3 & \leq 20 & \textit{finishing} \\
y_3 \leftrightarrow & 2x_1 + & 1.5x_2 + & 0.5x_3 & \leq 8 & \textit{carpentry}
\end{array}$$

$$x_1, x_2, x_3 \geq 0$$

$$\max \quad W = 48y_1 + 20y_2 + 8y_3$$

$$\begin{array}{lllll}
48y_1 + & 20y_2 + & 8y_3 & \geq 60 & \textit{desk} \\
6y_1 + & 2y_2 + & 1.5y_3 & \geq 30 & \textit{table} \\
y_1 + & 1.5y_2 + & 0.5y_3 & \geq 20 & \textit{chair}
\end{array}$$

$$y_1, y_2, y_3 \geq 0$$

INTERPRETATION OF THE DUAL PROBLEM

- An entrepreneur wishes to purchase all of Dakota's resources
- He needs, therefore, to determine the prices to pay for each unit of each resource

$$y_1 = \textit{price paid per ft of lumber board}$$

$$y_2 = \textit{price paid per h of finishing labor}$$

$$y_3 = \textit{price paid per h of carpentry labor}$$

- We solve the Dakota dual problem to determine y_1, y_2 and y_3

INTERPRETATION OF THE DUAL PROBLEM

- To convince Dakota to sell the raw resources, the resource prices must be set sufficiently high
- For example, the entrepreneur must offer Dakota at least $\$ \, 60$ for a combination of resources that consists of $8 \, ft$ of lumber board, $4 \, h$ of finishing and $2 \, h$ of carpentry, since Dakota could use this combination to sell a desk for $\$ \, 60$; this requires es the following dual constraint:

$$8 y_1 + 4 y_2 + 2 y_3 \geq 60$$

INTERPRETATION OF DUAL PROBLEM

- In the same way, we obtain the two additional constraints for a table and for a chair
- The i^{th} primal variable is associated with the i^{th} constraint in the dual problem statement
- The j^{th} dual variable is associated with the j^{th} constraint in the primal problem statement

EXAMPLE 3: DIET PROBLEM

- A new diet requires that all food eaten come from one of the four "*basic food groups*":
  - chocolate cake
  - soda
  - ice cream
  - cheesecake
- The four foods available for consumption are
  - brownie
  - cola
  - chocolate ice cream
  - pineapple cheesecake

EXAMPLE 3: DIET PROBLEM

- Minimum requirements for each day are:
  - 500 *cal*
  - 6 *oz* chocolate
  - 10 *oz* sugar
  - 8 *oz* fat
- The objective is to minimize the diet costs

EXAMPLE 3: DIET PROBLEM

food	*calories*	*chocolate (oz)*	*sugar (oz)*	*fat (oz)*	*costs (cents)*
brownie	**400**	**3**	**2**	**2**	**50**
chocolate ice cream (scoop)	**200**	**2**	**2**	**4**	**20**
cola (bottle)	**150**	***0***	**4**	**1**	**30**
pineapple cheesecake (piece)	**500**	***0***	**4**	**5**	**80**

PROBLEM FORMULATION

- Objective of the problem is to minimize the total costs of the diet
- Decision variables are defined for each day's purchases

$$x_1 = \textit{number of brownies}$$

$$x_2 = \textit{number of chocolate ice cream scoops}$$

$$x_3 = \textit{number of bottles of soda}$$

$$x_4 = \textit{number of pineapple cheesecake pieces}$$

PROBLEM FORMULATION

- The problem statement is

$$\min \quad Z = 50x_1 + 20x_2 + 30x_3 + 80x_4$$

$$s.t.$$

$$\begin{aligned}
400x_1 + 200x_2 + 150x_3 + 500x_4 &\geq 500 \; cal \\
3x_1 + 2x_2 &\geq 6 \; oz \\
2x_1 + 2x_2 + 4x_3 + 4x_4 &\geq 10 \; oz \\
2x_1 + 4x_2 + x_3 + 5x_4 &\geq 8 \; oz
\end{aligned}$$

$$x_i \geq 0 \qquad i = 1,4$$

EXAMPLE 3: DIET PROBLEM

- The dual problem is

$$\max \quad W = 500\, y_1 + 6\, y_2 + 10\, y_3 + 8\, y_4$$

$$s.t.$$

$$\begin{array}{lllll} 400\, y_1 + 3\, y_2 + 2\, y_3 + 2\, y_4 & \leq & 50 & & \textit{brownie} \\ 200\, y_1 + 2\, y_2 + 2\, y_3 + 4\, y_4 & \leq & 20 & & \textit{ice cream} \\ 150\, y_1 \qquad\quad\; + 4\, y_3 + \; y_4 & \leq & 30 & & \textit{soda} \\ 500\, y_1 \qquad\quad\; + 4\, y_3 + 5 y_4 & \leq & 80 & & \textit{cheesecake} \end{array}$$

$$y_1 , y_2 , y_3 , y_4 \geq 0$$

INTERPRETATION OF THE DUAL

- We consider a salesperson of "nutrients" who is interested in assuming that each dieter meets daily requirements by purchasing calories, sugar, fat and chocolate as "goods"
- The decision is to determine the prices charged

  y_i = *price per unit of required nutrient to sell to dieters*
- Objective of the salesperson is to set the prices y_i so as to maximize revenues from selling to the dieter the daily ration of required nutrients

INTERPRETATION OF DUAL

- Now, the dieter can purchase a brownie for 50 ¢ and have 400 *cal*, 3 *oz* of chocolate, 2 *oz* of sugar and 2 *oz* of fat
- The sales price y_i must be set sufficiently low to entice the buyer to get the required nutrients from the brownie:

$$400 y_1 + 3 y_2 + 2 y_3 + 2 y_4 \leq 50 \longleftarrow \textit{brownie constraint}$$

- We derive similar constraints for the ice cream, the soda and the cheesecake

DUAL PROBLEMS

$$\left.\begin{aligned} &\max && Z = \underline{c}^{T}\underline{x} \\ &s.t. && \\ & && \underline{A}\,\underline{x} \leq \underline{b} \\ & && \underline{x} \geq \underline{0} \end{aligned}\right\} (P)$$

$$\left.\begin{aligned} &\min && W = \underline{b}^{T}\underline{y} \\ &s.t. && \\ & && \underline{A}^{T}\underline{y} \geq \underline{c} \\ & && \underline{y} \geq \underline{0} \end{aligned}\right\} (D)$$

WEAK DUALITY THEOREM

- For any $\underline{x}$ feasible for (P) and any $\underline{y}$ feasible for (D), the following relation is satisfied

$$\underline{c}^T \underline{x} \leq \underline{b}^T \underline{y} \qquad \square$$

- Proof:

$$\underline{A}^T \underline{y} \geq \underline{c} \;\Rightarrow\; \underline{c}^T \leq \underline{y}^T \underline{A} \;\Rightarrow\; \underline{c}^T \underline{x} \leq \underline{y}^T \underline{A}\,\underline{x}$$

$$\underline{c}^T \underline{x} \leq \underline{y}^T \underline{A}\,\underline{x} \leq \underline{y}^T \underline{b} = \underline{b}^T \underline{y} \qquad \square$$

COROLLARY 1 OF THE WEAK DUALITY THEOREM

$$\underline{x} \;\; is \;\; feasible \;\; for \left(P\right) \;\; \Rightarrow \;\; \underline{c}^{\,T} \underline{x} \;\; \leq \;\; \underline{y}^{\,T} \underline{b}$$

$$for \;\; any \;\; feasible \;\; \underline{y} \;\; for \left(D\right)$$

$$\underline{c}^{\,T} \underline{x} \;\; \leq \;\; \underline{y}^{\,*T} \underline{b} \;\; = \;\; min\, W$$

$$\boxed{\begin{array}{c} for \;\; any \;\; feasible \;\; \underline{x} \;\; for \left(P\right), \\ \underline{c}^{\,T} \underline{x} \;\; \leq \;\; min\, W \end{array}}$$

COROLLARY 2 OF THE WEAK DUALITY THEOREM

$$\underline{y} \text{ is feasible for } (D) \;\Rightarrow\; \underline{c}^T \underline{x} \;\leq\; \underline{y}^T \underline{b}$$

$$\text{for every feasible } \underline{x} \text{ for } (P)$$

$$\max Z \;=\; \max \underline{c}^T \underline{x} \;=\; \underline{c}^T \underline{x}^* \;\leq\; \underline{y}^T \underline{b}$$

$$\boxed{\begin{array}{c} \text{for any feasible } \underline{y} \text{ of } (D), \\ \underline{y}^T \underline{b} \;\geq\; \max Z \end{array}}$$

COROLLARIES 3 AND 4 OF THE WEAK DUALITY THEOREM

If (P) is feasible and $max\ Z$ is unbounded, i.e.,

$$Z \rightarrow +\infty ,$$

then, (D) has no feasible solution.

If (D) is feasible and $min\ Z$ is unbounded, i.e.,

$$Z \rightarrow -\infty ,$$

then, (P) is infeasible.

DUALITY THEOREM APPLICATION

- Consider the maximization problem

$$\left.\begin{array}{l} \max Z = x_1 + 2x_2 + 3x_3 + 4x_4 = \underbrace{\begin{bmatrix} 1, 2, 3, 4 \end{bmatrix}}_{\underline{c}^T} \underline{x} \\ s.t. \\ \qquad \underbrace{\begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 1 & 3 & 2 \end{bmatrix}}_{\underline{A}} \underline{x} \le \underbrace{\begin{bmatrix} 20 \\ 20 \end{bmatrix}}_{\underline{b}} \\ \qquad \underline{x} \ge \underline{0} \end{array}\right\} (P)$$

DUALITY THEOREM APPLICATION

- The corresponding dual is given by

$$\left.\begin{array}{lll} \min & W = \underline{b}^T \underline{y} & \\ s.t. & & \\ & \underline{A}^T \underline{y} \geq \underline{c} & \\ & \underline{y} \geq \underline{0} & \end{array}\right\} \quad (D)$$

- With the appropriate substitutions, we obtain

DUALITY THEOREM APPLICATION

$$\min \qquad W = 20\,y_1 + 20\,y_2$$

$$s.t.$$

$$
\begin{aligned}
y_1 + 2y_2 &\ge 1 \\
2y_1 + y_2 &\ge 2 \\
2y_1 + 3y_2 &\ge 3 \\
3y_1 + 2y_2 &\ge 4 \\
y_1 \ge 0, \quad y_2 &\ge 0
\end{aligned}
$$

GENERALIZED FORM OF THE DUAL

- Consider the primal decision

$$x_i = 1, \quad i = 1, 2, 3, 4 \,;$$

decision is feasible for (P) with

$$Z = \underline{c}^T \underline{x} = 10$$

- The dual decision

$$y_i = 1, \quad i = 1, 2$$

is feasible for (D) with

$$W = \underline{b}^T \underline{y} = 40$$

DUALITY THEOREM APPLICATION

- Clearly,

$$Z\left(x_1, x_2, x_3, x_4\right) = 10 \;\leq\; 40 = W\left(y_1, y_2\right)$$

and so clearly, the feasible decision for (P) and (D)

satisfy the *Weak Duality Theorem*

- Moreover, we have

$$\textit{corollary}\ 1 \Rightarrow 10 \leq \min W = W\left(y_1^*, y_2^*\right)$$

$$\textit{corollary}\ 2 \Rightarrow \max Z = Z\left(x_1^*, x_2^*, x_3^*, x_4^*\right) \leq \underline{b}^T \underline{y} = 40$$

COROLLARIES 5 AND 6

(P) is feasible and (D) is infeasible, then,

(P) is unbounded $\square$

(D) is feasible and (P) is infeasible, then,

(D) is unbounded $\square$

EXAMPLE

- Consider the primal dual problems:

$$\left.\begin{aligned} &\max Z = x_1 + x_2 \\ &s.t. \\ &\quad -x_1 + x_2 + x_3 \leq 2 \\ &\quad -2x_1 + x_2 - x_3 \leq 1 \\ &\quad x_1, x_2, x_3 \geq 0 \end{aligned}\right\} (P) \qquad \left.\begin{aligned} &\min W = 2y_1 + y_2 \\ &s.t. \\ &\quad -y_1 - 2y_2 \geq 1 \\ &\quad y_1 + y_2 \geq 1 \\ &\quad y_1 - y_2 \geq 0 \\ &\quad y_1, y_2 \geq 0 \end{aligned}\right\} (D)$$

- Now

$$\underline{x} = \underline{0} \text{ is feasible for } (P)$$

EXAMPLE

$$\underline{x} = \underline{0} \ \textit{ is feasible for } (P)$$

but

$$-\ y_1 - 2y_2 \geq 1$$

is impossible for (D) since it is inconsistent with

$$y_1, y_2 \geq 0$$

- Since (D) is infeasible, it follows from Corollary 5 that $Z \rightarrow \infty$
- You are able to show this result by solving (P) using the simplex scheme

OPTIMALITY CRITERION THEOREM

- We consider the primal-dual problems (P) and (D) with

$$\left.\begin{array}{l} \underline{x}^{\,0} \text{ is feasible for } (P) \\ \underline{y}^{\,0} \text{ is feasible for } (D) \\ \underline{c}^{T}\underline{x}^{\,0} = \underline{b}^{T}\underline{y}^{\,0} \end{array}\right\} \Rightarrow \begin{array}{l} \underline{x}^{\,0} \text{ is optimal for } (P) \\ \text{and} \\ \underline{y}^{\,0} \text{ is optimal for } (D) \end{array}$$

- We next provide the proof:

$$\begin{array}{l} \underline{x}^{\,0} \text{ is feasible for } (P) \\ \underline{y}^{\,0} \text{ is feasible for } (D) \end{array} \quad \underset{Theorem}{\overset{Weak\ \ Duality}{\Longrightarrow}} \quad \underline{c}^{T}\underline{x}^{\,0} \leq \underline{b}^{T}\underline{y}^{\,0}$$

OPTIMALITY CRITERION THEOREM

but we are given that

$$\underline{c}^{T}\,\underline{x}^{0} = \underline{b}^{T}\,\underline{y}^{0}$$

and so it follows that $\forall$ *feasible* $\underline{x}$ *with* $\underline{y}^{0}$ *feasible*

$$\underline{c}^{T}\,\underline{x} \leq \underline{b}^{T}\,\underline{y}^{0} = \underline{c}^{T}\,\underline{x}^{0}$$

and so $\underline{x}^{0}$ is *optimal* ;

similarly, $\forall$ *feasible* $\underline{y}$ *with* $\underline{x}^{0}$ *feasible*

$$\underline{b}^{T}\,\underline{y} \geq \underline{c}^{T}\,\underline{x}^{0} = \underline{b}^{T}\,\underline{y}^{0}$$

and so it follows that $\underline{y}^{0}$ is *optimal*

MAIN DUALITY THEOREM

(P) is feasible and (D) is feasible; then,

$\exists\ \underline{x}^{*}$ *feasible for* (P) *which is optimal and*

$\exists\ \underline{y}^{*}$ *feasible for* (D) *which is optimal such that*

$$\underline{c}^{T}\underline{x}^{*} = \underline{b}^{T}\underline{y}^{*}$$

COMPLEMENTARY SLACKNESS CONDITIONS

- $\underline{x}^*$ and $\underline{y}^*$ are optimal for (P) and (D) respectively, if and only if

$$0 = \left(\underline{y}^{*T}\underline{A} - \underline{c}^T\right)\underline{x}^* + \underline{y}^{*T}\left(\underline{b} - \underline{A}\,\underline{x}^*\right)$$

$$= \underline{y}^{*T}\underline{b} - \underline{c}^T\underline{x}^*$$

- We prove this equivalence result by defining the slack variables $\underline{u} \in \mathbb{R}^m$ and $\underline{v} \in \mathbb{R}^n$ such that $\underline{x}$ and $\underline{y}$ are feasible; at the optimum,

$$\underline{A}\,\underline{x}^* + \underline{u}^* = \underline{b} \qquad \underline{x}^*, \underline{u}^* \geq \underline{0}$$

$$\underline{A}^T\underline{y}^* - \underline{v}^* = \underline{c} \qquad \underline{y}^*, \underline{v}^* \geq \underline{0}$$

COMPLEMENTARY SLACKNESS CONDITIONS

where the optimal values of the slack variables

$\underline{u}^{*}$ **and** $\underline{v}^{*}$ **depend on the optimal values**

$\underline{x}^{*}$ **and** $\underline{y}^{*}$

- **Now,**

$$\underline{y}^{*T}\underline{A}\underline{x}^{*} + \underline{y}^{*T}\underline{u}^{*} = \underline{y}^{*T}\underline{b} = \underline{b}^{T}\underline{y}^{*}$$

$$\underbrace{\underline{x}^{*T}\underline{A}^{T}\underline{y}^{*}}_{\underline{y}^{*T}\underline{A}\,\underline{x}^{*}} - \underline{x}^{*T}\underline{v}^{*} = \underline{x}^{*T}\underline{c} = \underline{c}^{T}\underline{x}^{*}$$

COMPLEMENTARY SLACKNESS CONDITIONS

- This implies that

$$\underline{y}^{*T}\underline{u}^{*} + \underline{v}^{*T}\underline{x}^{*} = \underline{b}^{T}\underline{y}^{*} - \underline{c}^{T}\underline{x}^{*}$$

- We need to prove optimality which is true if and

  only if

$$\underline{y}^{*T}\underline{u}^{*} + \underline{v}^{*T}\underline{x}^{*} = 0$$

COMPLEMENTARY SLACKNESS CONDITIONS

- However,

$$\underline{x}^*, \underline{y}^* \textit{ are optimal}$$

$$\xRightarrow[\textit{Duality Theorem}]{\textit{Main}}$$

$$\underline{c}^T \underline{x}^* = \underline{b}^T \underline{y}^* \Rightarrow \underline{y}^{*T} \underline{u}^* + \underline{v}^{*T} \underline{x}^* = 0$$

- Also,

$$\underline{y}^{*T} \underline{u}^* + \underline{v}^{*T} \underline{x}^* = 0 \Rightarrow \underline{b}^T \underline{y}^* = \underline{c}^T \underline{x}^*$$

$$\xRightarrow[\textit{Criterion Theorem}]{\textit{Optimality}}$$

$$\underline{x}^* \textit{ is optimal for } (P) \textit{ and } \underline{y}^* \textit{ is optimal for } (D)$$

COMPLEMENTARY SLACKNESS CONDITIONS

- Note that

$$\underline{x}^{*}, \underline{y}^{*}, \underline{u}^{*}, \underline{v}^{*} > 0 \Rightarrow \textit{component-wise each element} \geq 0$$

$$\underline{y}^{*T}\underline{u}^{*} + \underline{v}^{*}\underline{x}^{*} = 0 \Rightarrow y_i^{*}u_i^{*} = 0 \;\; \forall i = 1, \ldots, m$$

$$\textit{and } v_j^{*}x_j^{*} = 0 \;\; \forall j = 1, \ldots, n$$

- At the optimum,

$$y_i^{*}\left(b_i - \sum_{j=1}^{n} a_{ij}x_j^{*}\right) = 0 \quad i = 1, \ldots, m$$

and

$$x_j^{*}\left(\sum_{i=1}^{m} a_{ji}y_i^{*} - c_j\right) = 0 \quad j = 1, \ldots, n$$

COMPLEMENTARY SLACKNESS CONDITIONS

- Hence, for $i = 1, 2, \ldots, m$

$$y_i^* > 0 \Rightarrow b_i = \sum_{j=1}^{n} a_{ij} x_j^*$$

and

$$b_i - \sum_{j=1}^{m} a_{ij} x_i^* > 0 \Rightarrow y_i^* = 0$$

- Similarly for $j = 1, 2, \ldots, n$

$$x_i^* > 0 \Rightarrow \sum_{i=1}^{m} a_{ji} y_i^* = c_j$$

and

$$\sum_{i=1}^{m} a_{ji} y_i^* - c_j > 0 \Rightarrow x_j^* = 0$$

EXAMPLE

$$\left.\begin{array}{ll} \max & Z = x_1 + 2x_2 + 3x_3 + 4x_4 \\ \\ s.t. & \\ \\ & x_1 + 2x_2 + 2x_3 + 3x_4 \leq 20 \\ \\ & 2x_1 + x_2 + 3x_3 + 2x_4 \leq 20 \\ \\ & x_i \geq 0 \quad i = 1, \ldots, 4 \end{array}\right\} (P)$$

EXAMPLE

$$\left.\begin{aligned}
\min \quad & W = 20y_1 + 20y_2 \\
s.t. \quad & \\
& y_1 + 2y_2 \geq 1 \\
& 2y_1 + y_2 \geq 2 \\
& 2y_1 + 3y_2 \geq 3 \\
& 3y_1 + 2y_2 \geq 4 \\
& y_1, y_2 \geq 0
\end{aligned}\right\} \quad (D)$$

EXAMPLE

$\underline{x}^*, \underline{y}^*$ *optimal* $\Rightarrow$

$$y_1^*\left(20 - x_1^* - 2x_2^* - 2x_3^* - 3x_4^*\right) = 0$$

$$y_2^*\left(20 - 2x_1^* - x_2^* - 3x_3^* - 2x_4^*\right) = 0$$

$\underline{y}^* = \begin{bmatrix} 1.2 \\ 0.2 \end{bmatrix}$ **is given as an optimal solution with**

$$min\ W = 28$$

EXAMPLE

$$x_1^* + 2x_2^* + 2x_3^* + 3x_4^* = 20$$

$$2x_1^* + x_2^* + 3x_3^* + 2x_4^* = 20$$

$$y_1^* + 2y_2^* = 1.2 + 0.4 > 1 \Rightarrow x_1^* = 0$$

$$2y_1^* + y_2^* = 2.4 + 0.2 > 2 \Rightarrow x_2^* = 0$$

$$2y_1^* + 3y_2^* = 2.4 + 0.6 = 3$$

$$3y_1^* + 2y_2^* = 3.6 + 0.4 = 4$$

so that

$$\left.\begin{array}{ll} 2x_3^* + 3x_4^* = 20 & \Rightarrow x_3^* = 4 \\ 3x_3^* + 2x_4^* = 20 & \Rightarrow x_4^* = 4 \end{array}\right\}$$

COMPLEMENTARY SLACKNESS CONDITION APPLICATIONS

- Key uses of *c.s. conditions* are
  - finding optimal (P) solution given optimal (D) solution and vice versa
  - verification of optimality of solution (whether a feasible solution is optimal)
- We can start with a feasible solution and attempt to construct an optimal dual solution; if we succeed, then the feasible primal solution is *optimal*

DUALITY

$$\left.\begin{aligned} \max \quad & Z = \underline{c}^{T}\underline{x} \\ \text{s.t.} \quad & \\ & \underline{A}\,\underline{x} \leq \underline{b} \\ & \underline{x} \geq \underline{0} \end{aligned}\right\} (P)$$

$$\left.\begin{aligned} \min \quad & W = \underline{b}^{T}\underline{y} \\ \text{s.t.} \quad & \\ & \underline{A}^{T}\underline{y} \geq \underline{c} \\ & \underline{y} \geq \underline{0} \end{aligned}\right\} (D)$$

DUALITY

- Suppose the primal problem is minimization, then,

$$\min \quad Z = \underline{c}^T \underline{x} \qquad (P)$$

$$s.t.$$

$$\underline{A}\,\underline{x} \geq \underline{b}$$

$$\underline{x} \geq \underline{0}$$

$$\max \quad W = \underline{b}^T \underline{y}$$

$$s.t. \qquad (D)$$

$$\underline{A}^T \underline{y} \leq \underline{c}$$

$$\underline{y} \geq \underline{0}$$

INTERPRETATION

- The economic interpretation is

$$Z^{*} = \max Z = \underline{c}^{T}\underline{x}^{*} = \underline{b}^{T}\underline{y}^{*} = W^{*} = \min W$$

$$\left.\begin{array}{l} b_i - \textit{constrained resource quantities} \\ y^{*}_{i} - \textit{optimal dual variables} \end{array}\right\} i = 1, 2, \dots, m$$

- Suppose, we change

$$b_i \rightarrow b_i + \Delta b_i \Rightarrow \Delta Z = y^{*}_{i} \Delta b_i$$

- In words, the optimal dual variable for each primal constraint gives the net change in the optimal value of the objective function Z for a one unit change in the constraint on resources

INTERPRETATION

- Economists refer to the dual variable as the *shadow price* on the constraint resource
- The *shadow price* determines the value/worth of having an additional quantity of a resource
- In the previous example, the optimal dual variables indicate that the worth of another unit of resource 1 is 1.2 while that of another unit of resource 2 is 0.2

GENERALIZED FORM OF THE DUAL

- We start out with

$$\left.\begin{array}{lrcl} \max & Z & = & \underline{c}^{T}\,\underline{x} \\ s.t. & & & \\ & \underline{A}\,\underline{x} & = & \underline{b} \\ & \underline{x} & \geq & \underline{0} \end{array}\right\} (P)$$

GENERALIZED FORM OF THE DUAL

- To find (D), we first put (P) in symmetric form

$$\begin{array}{lcrcr} \underline{y}_{+} & \leftrightarrow & \underline{A}\underline{x} & \leq & \underline{b} \\ \underline{y}_{-} & \leftrightarrow & -\underline{A}\underline{x} & \leq & -\underline{b} \\ & & \underline{x} & \geq & \underline{0} \end{array} \qquad \begin{bmatrix} \underline{A} \\ \\ -\underline{A} \end{bmatrix} \underline{x} \leq \begin{bmatrix} \underline{b} \\ \\ -\underline{b} \end{bmatrix}$$

symmetric form

GENERALIZED FORM OF THE DUAL

- Let

$$\underline{y} = \underline{y}_{+} - \underline{y}_{-}$$

- We rewrite the problem as

$$\min W = \underline{b}^{T} \underline{y}$$

$$s.t.$$

$$\underline{A}^{T} \underline{y} \geq \underline{c}$$

$$\underline{y} \text{ is unsigned}$$

- The $c.s.$ conditions apply

$$\underline{x}^{*T} \left(\underline{A}^{T} \underline{y}^{*} - \underline{c} \right) = \underline{0}$$

EXAMPLE 5: THE PRIMAL

$$\max Z = x_1 - x_2 + x_3 - x_4$$

s.t.

$$
\begin{array}{lcl}
y_1 & \leftrightarrow & x_1 + x_2 + x_3 + x_4 = 8 \\
y_2 & \leftrightarrow & x_1 \leq 8 \\
y_3 & \leftrightarrow & x_2 \leq 4 \\
y_4 & \leftrightarrow & -x_2 \leq 4 \\
y_5 & \leftrightarrow & x_3 \leq 4 \\
y_6 & \leftrightarrow & -x_3 \leq 2 \\
y_7 & \leftrightarrow & x_4 \leq 10 \\
 & & x_1, x_4 \geq 0 \\
 & & x_2, x_3 \text{ unsigned}
\end{array}
\qquad (P)
$$

EXAMPLE 5: THE DUAL

$$\min W = 8y_1 + 8y_2 + 4y_3 + 4y_4 + 4y_5 + 2y_6 + 10y_7$$

s.t.

$$\left.\begin{array}{lllllllllll}
x_1 \leftrightarrow & y_1 & + y_2 & & & & & & \geq & 1 \\
x_2 \leftrightarrow & y_1 & & + y_3 & - y_4 & & & & = & -1 \\
x_3 \leftrightarrow & y_1 & & & & + y_5 & - y_6 & & = & 1 \\
x_4 \leftrightarrow & y_1 & & & & & & + y_7 & \geq & -1
\end{array}\right\} (D)$$

$$y_2, \ldots\ldots, y_7 \geq 0$$

$$y_1 \text{ unsigned}$$

EXAMPLE 5: *c.s. conditions*

- **We are given that**

$$\underline{x}^{*} = \begin{bmatrix} 8 \\ -4 \\ 4 \\ 0 \end{bmatrix}$$

  is optimal for (P)

- **Then the *c.s. conditions* obtain**

$$x_1^{*}\left(y_1^{*} + y_2^{*} - 1 \right) = 0$$

EXAMPLE 5: *c.s. conditions*

so that

$$x_1^* = 8 > 0 \Rightarrow y_1^* + y_2^* = 1$$

and so $y_2^* = 1 - y_1^*$

- **The other** ***c.s. conditions*** **require**

$$y_i^* \left(\sum_{j=1}^{4} a_{ij} x_j^* - b_i \right) = 0$$

- **Now,** $x_4^* = 0$ **implies** $x_4^* - 10 < 0$ **and so** $y_7^* = 0$

EXAMPLE 5: *c.s. conditions*

- Also, $x_3^* = 4$ implies

$$y_6^* = 0$$

- We similarly use the *c.s. conditions*

$$x_j^* \left(\sum_{i=1}^{7} a_{ji}\, y_i^* - c_j \right) = 0$$

to provide implications on the y_i^* variables

EXAMPLE 5: *c.s. conditions*

- Since $x_2^* = -4$, then we have

$$y_3^* = 0$$

- Now, with $y_7^* = 0$ we have

$$y_1^* > -1$$

- Now, we have already shown that

$$y_2^* = 1 - y_1^*$$

EXAMPLE 5

- Suppose that

$$y_1^* = 1$$

and so,

$$y_2^* = 0$$

- Furthermore,

$$y_1^* + y_3^* - y_4^* = 1 - y_4^* = -1$$

implies that

$$y_4^* = 2$$

EXAMPLE 5

- Also

$$y_1^* + y_5^* - y_6^* = 1$$

implies

$$1 + y_5^* = 1$$

and so

$$y_5^* = 0$$

EXAMPLE 5

- Therefore, as $W = \underline{b}^T \underline{y}$

$$W\left(\underline{y}^*\right) = (8)(1)+(8)(0)+(4)(0)+(4)(2)+$$

$$(4)(0)+(2)(0)+(10)(0)$$

$$= 16$$

and so

$W^* = 16 = Z^* \Leftrightarrow$ optimality of (P) and (D)

PRIMAL – DUAL TABLE

primal (maximize)	*dual* (minimize)
$\underline{A}$ (coefficient matrix)	$\underline{A}^T$ (transpose of the coefficient matrix)
$\underline{b}$ (right-hand side vector)	$\underline{b}$ (cost vector)
$\underline{c}$ (price vector)	$\underline{c}$ (right hand side vector)
i^{th} constraint is $=$ type	the dual variable y_i is unrestricted in sign
i^{th} constraint is $\leq$ type	the dual variable $y_i \geq 0$
i^{th} constraint is $\geq$ type	the dual variable $y_i \leq 0$
x_j is unrestricted	j^{th} dual constraint is $=$ type
$x_j \geq 0$	j^{th} dual constraint is $\geq$ type
$x_j \leq 0$	j^{th} dual constraint is $\leq$ type