ECE 307- Techniques for Engineering Decisions

Lecture 4. Duality Concepts in Linear Programming

George Gross

Department of Electrical and Computer Engineering

University of Illinois at Urbana-Champaign
Definition: A *LP* is in *symmetric form* if all the variables are restricted to be *nonnegative* and all the constraints are inequalities of the type:

<table>
<thead>
<tr>
<th>objective type</th>
<th>corresponding inequality type</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>max</em></td>
<td>≤</td>
</tr>
<tr>
<td><em>min</em></td>
<td>≥</td>
</tr>
</tbody>
</table>
We first define the **primal** and **dual** problems

\[
\begin{align*}
\text{max} & \quad Z = c^T x \\
\text{s.t.} & \quad Ax \leq b \\
& \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{min} & \quad W = b^T y \\
\text{s.t.} & \quad A^T y \geq c \\
& \quad y \geq 0
\end{align*}
\]
The problems \((P)\) and \((D)\) are called the symmetric dual LP problems; we restate them as

\[
\begin{align*}
\text{max } Z &= c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \\
\text{s.t.} \\
a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n &\leq b_1 \\
a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n &\leq b_2 \\
& \vdots \\
a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n &\leq b_m \\
x_1 &\geq 0, \quad x_2 \geq 0, \quad \ldots, \quad x_n \geq 0
\end{align*}
\]
DUALITY DEFINITIONS

\[ \min W = b_1 y_1 + b_2 y_2 + \ldots + b_m y_m \]

s.t.

\[
\begin{align*}
    a_{11} y_1 + a_{21} y_2 + \ldots + a_{m1} y_m & \geq c_1 \\
    a_{12} y_1 + a_{22} y_2 + \ldots + a_{m2} y_m & \geq c_2 \\
    & \vdots \\
    a_{1n} y_1 + a_{2n} y_2 + \ldots + a_{mn} y_m & \geq c_n
\end{align*}
\]

\[ y_1 \geq 0, \quad y_2 \geq 0, \quad \ldots, \quad y_m \geq 0 \]
**EXAMPLE 1: MANUFACTURER TRANSPORTATION PROBLEM**

*shipment cost coefficients*

<table>
<thead>
<tr>
<th></th>
<th>$R_1$</th>
<th>$R_2$</th>
<th>$R_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_1$</td>
<td>2</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>$W_2$</td>
<td>5</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

![Diagram showing the transportation problem with warehouses $W_1$, $W_2$, and retail stores $R_1$, $R_2$, $R_3$.]
EXAMPLE 1: MANUFACTURER TRANSPORTATION PROBLEM

- We are given that the supplies stored in warehouses $W_1$ and $W_2$ satisfy:
  
  \begin{align*}
  \text{supply at } W_1 & \leq 300 \\
  \text{supply at } W_2 & \leq 600
  \end{align*}

- We are also given the demands needed to be met at the retail stores $R_1$, $R_2$, and $R_3$:
  
  \begin{align*}
  \text{demand at } R_1 & \geq 200 \\
  \text{demand at } R_2 & \geq 300 \\
  \text{demand at } R_3 & \geq 400
  \end{align*}
EXAMPLE 1: MANUFACTURER TRANSPORTATION PROBLEM

- The problem is to determine the least-cost shipping schedule.

- We define the decision variable

\[ x_{ij} = \text{quantity shipped from } W_i \text{ to } R_j \quad i = 1, 2, \quad j = 1, 2, 3 \]

- The shipping costs may be viewed as

\[ c_{ij} = \text{element } i, j \text{ of the transportation cost matrix} \]
\[
\text{FORMULATION STATEMENT}
\]

\[
\begin{align*}
\min \ Z & = \sum_{i=1}^{2} \sum_{j=1}^{3} c_{ij} \ x_{ij} = 2x_{11} + 4x_{12} + 3x_{13} + 5x_{21} + 3x_{22} + 4x_{23} \\
\text{s.t.} & \quad x_{11} + x_{12} + x_{13} \leq 300 \\
& \quad x_{21} + x_{22} + x_{23} \leq 600 \\
& \quad x_{11} + x_{21} \geq 200 \\
& \quad x_{12} + x_{22} \geq 300 \\
& \quad x_{13} + x_{23} \geq 400 \\
& \quad x_{ij} \geq 0 \quad i = 1, 2, \quad j = 1, 2, 3
\end{align*}
\]
DUAL PROBLEM SETUP USING SYMMETRIC FORM

\[
\min Z = \sum_{i=1}^{2} \sum_{j=1}^{3} c_{ij} x_{ij}
\]

s.t.

\[
y_1 \leftrightarrow -x_{11} - x_{12} - x_{13} \geq -300
\]

\[
y_2 \leftrightarrow -x_{21} - x_{22} - x_{23} \geq -600
\]

\[
y_3 \leftrightarrow x_{11} + x_{21} \geq 200
\]

\[
y_4 \leftrightarrow x_{12} + x_{22} \geq 300
\]

\[
y_5 \leftrightarrow x_{13} + x_{23} \geq 400
\]

\[
x_{ij} \geq 0 \quad i = 1, 2 \quad j = 1, 2, 3
\]
DUAL PROBLEM SETUP

\[
\text{max} \ W = -300y_1 - 600y_2 + 200y_3 + 300y_4 + 400y_5
\]

s.t.

\[
\begin{align*}
- y_1 & + y_3 & \leq & c_{11} = 2 \\
- y_1 & + y_4 & \leq & c_{12} = 4 \\
- y_1 & + y_5 & \leq & c_{13} = 3 \\
- y_2 & + y_3 & \leq & c_{21} = 5 \\
- y_2 & + y_4 & \leq & c_{22} = 3 \\
- y_2 & + y_5 & \leq & c_{23} = 4 \\
y_i & \geq 0 & i = 1, 2, \ldots, 5
\end{align*}
\]
The moving company proposes to the manufacturer to:

- buy all the 300 units at $W_1$ at $y_1 / \text{unit}$
- buy all the 600 units at $W_2$ at $y_2 / \text{unit}$
- sell all the 200 units at $R_1$ at $y_3 / \text{unit}$
- sell all the 300 units at $R_2$ at $y_4 / \text{unit}$
- sell all the 400 units at $R_3$ at $y_5 / \text{unit}$

To convince the manufacturer to get the business, the mover ensures that the delivery fees cannot exceed the transportation costs the manufacturer would incur (the dual constraints).
THE DUAL PROBLEM INTERPRETATION

- \( y_1 \) + \( y_3 \) \( \leq \) \( c_{11} = 2 \)
- \( y_1 \) + \( y_4 \) \( \leq \) \( c_{12} = 4 \)
- \( y_1 \) + \( y_5 \) \( \leq \) \( c_{13} = 3 \)
- \( y_2 \) + \( y_3 \) \( \leq \) \( c_{21} = 5 \)
- \( y_2 \) + \( y_4 \) \( \leq \) \( c_{22} = 3 \)
- \( y_2 \) + \( y_5 \) \( \leq \) \( c_{23} = 4 \)

The mover wishes to maximize profits, i.e.,

revenues - costs \( \Rightarrow \) dual cost objective function

\[ \text{max} W = -300 \ y_1 - 600 \ y_2 + 200 \ y_3 + 300 \ y_4 + 400 \ y_5 \]
EXAMPLE 2: FURNITURE PRODUCTS

Resource requirements

<table>
<thead>
<tr>
<th>item</th>
<th>sales price ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>desks</td>
<td>60</td>
</tr>
<tr>
<td>tables</td>
<td>30</td>
</tr>
<tr>
<td>chairs</td>
<td>20</td>
</tr>
</tbody>
</table>
The Dakota Furniture Company manufacturing:

<table>
<thead>
<tr>
<th>resource</th>
<th>desk</th>
<th>table</th>
<th>chair</th>
<th>available</th>
</tr>
</thead>
<tbody>
<tr>
<td>lumber board (ft)</td>
<td>8</td>
<td>6</td>
<td>1</td>
<td>48</td>
</tr>
<tr>
<td>finishing (h)</td>
<td>4</td>
<td>2</td>
<td>1.5</td>
<td>20</td>
</tr>
<tr>
<td>carpentry (h)</td>
<td>2</td>
<td>1.5</td>
<td>0.5</td>
<td>8</td>
</tr>
</tbody>
</table>

We assume that the demand for desks, tables and chairs is unlimited and the two required resources – lumber and labor – are already purchased.

The decision problem is to maximize total revenues.
We define decision variables

\[ x_1 = \text{number of desks produced} \]
\[ x_2 = \text{number of tables produced} \]
\[ x_3 = \text{number of chairs produced} \]

The Dakota problem is

\[
\begin{align*}
\text{max} \quad Z &= 60x_1 + 30x_2 + 20x_3 \\
\text{s.t.} \quad &y_1 \leftrightarrow 8x_1 + 6x_2 + x_3 \leq 48 \quad \text{lumber} \\
&y_2 \leftrightarrow 4x_1 + 2x_2 + 1.5x_3 \leq 20 \quad \text{finishing} \\
&y_3 \leftrightarrow 2x_1 + 1.5x_2 + 0.5x_3 \leq 8 \quad \text{carpentry} \\
&x_1, x_2, x_3 \geq 0
\end{align*}
\]
The dual problem is

\[
\text{min } W = 48y_1 + 20y_2 + 8y_3
\]

\[
\text{s.t. }
\begin{align*}
8y_1 &+ 4y_2 + 2y_3 \geq 60 & \text{desk} \\
6y_1 &+ 2y_2 + 1.5y_3 \geq 30 & \text{table} \\
y_1 &+ 1.5y_2 + 0.5y_3 \geq 20 & \text{chair}
\end{align*}
\]

\[
y_1, y_2, y_3 \geq 0
\]
PRIMAL AND DUAL PROBLEM FORMULATION

\[ \begin{align*}
\text{max } Z &= 60x_1 + 30x_2 + 20x_3 \\
y_1 &\leftrightarrow 8x_1 + 6x_2 + x_3 \leq 48 \quad \text{lumber} \\
y_2 &\leftrightarrow 4x_1 + 2x_2 + 1.5x_3 \leq 20 \quad \text{finishing} \\
y_3 &\leftrightarrow 2x_1 + 1.5x_2 + 0.5x_3 \leq 8 \quad \text{carpentry} \\
x_1, x_2, x_3 &\geq 0
\end{align*} \]

\[ \begin{align*}
\text{max } W &= 48y_1 + 20y_2 + 8y_3 \\
48y_1 + 20y_2 + 8y_3 &\geq 60 \quad \text{desk} \\
6y_1 + 2y_2 + 1.5y_3 &\geq 30 \quad \text{table} \\
y_1 + 1.5y_2 + 0.5y_3 &\geq 20 \quad \text{chair} \\
y_1, y_2, y_3 &\geq 0
\end{align*} \]
INTERPRETATION OF THE DUAL PROBLEM

- An entrepreneur wishes to purchase all of Dakotas resources.

- He needs, therefore, to determine the prices to pay for each unit of each resource

\[
y_1 = \text{price paid per ft of lumber board}
\]

\[
y_2 = \text{price paid per h of finishing labor}
\]

\[
y_3 = \text{price paid per h of carpentry labor}
\]

- We solve the Dakota dual problem to determine \( y_1, y_2 \) and \( y_3 \)
To convince Dakota to sell the raw resources, the resource prices must be set sufficiently high.

For example, the entrepreneur must offer Dakota at least $60 for a combination of resources that consists of 8 ft of lumber board, 4 h of finishing and 2 h of carpentry, since Dakota could use this combination to sell a desk for $60; this requires the following dual constraint:

\[ 8y_1 + 4y_2 + 2y_3 \geq 60 \]
In the same way, we obtain the two additional constraints for a table and for a chair.

The $i^{th}$ primal variable is associated with the $i^{th}$ constraint in the dual problem statement.

The $j^{th}$ dual variable is associated with the $j^{th}$ constraint in the primal problem statement.
EXAMPLE 3: DIET PROBLEM

A new diet requires that all food eaten come from one of the four “basic food groups”:

- chocolate cake
- soda
- ice cream
- cheesecake

The four foods available for consumption are

- brownie
- cola
- chocolate ice cream
- pineapple cheesecake
EXAMPLE 3: DIET PROBLEM

- Minimum requirements for each day are:
  - 500 cal
  - 6 oz chocolate
  - 10 oz sugar
  - 8 oz fat

- The objective is to minimize the diet costs
### EXAMPLE 3: DIET PROBLEM

<table>
<thead>
<tr>
<th>food</th>
<th>calories</th>
<th>chocolate (oz)</th>
<th>sugar (oz)</th>
<th>fat (oz)</th>
<th>costs (cents)</th>
</tr>
</thead>
<tbody>
<tr>
<td>brownie</td>
<td>400</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>50</td>
</tr>
<tr>
<td>chocolate ice cream</td>
<td>200</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>(scoop)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>cola</td>
<td>150</td>
<td>0</td>
<td>4</td>
<td>1</td>
<td>30</td>
</tr>
<tr>
<td>(bottle)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>pineapple cheesecake</td>
<td>500</td>
<td>0</td>
<td>4</td>
<td>5</td>
<td>80</td>
</tr>
<tr>
<td>(piece)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Objective of the problem is to minimize the total costs of the diet

Decision variables are defined for each day’s purchases

\[ x_1 = \text{number of brownies} \]
\[ x_2 = \text{number of chocolate ice cream scoops} \]
\[ x_3 = \text{number of bottles of soda} \]
\[ x_4 = \text{number of pineapple cheesecake pieces} \]
The problem statement is

\[
\begin{align*}
\text{min} \quad & Z = 50 x_1 + 20 x_2 + 30 x_3 + 80 x_4 \\
\text{s.t.} \quad & 400 x_1 + 200 x_2 + 150 x_3 + 500 x_4 \geq 500 \text{ cal} \\
& 3 x_1 + 2 x_2 \geq 6 \text{ oz} \\
& 2 x_1 + 2 x_2 + 4 x_3 + 4 x_4 \geq 10 \text{ oz} \\
& 2 x_1 + 4 x_2 + x_3 + 5 x_4 \geq 8 \text{ oz} \\
& x_i \geq 0 \quad i = 1, 4
\end{align*}
\]
EXAMPLE 3: DIET PROBLEM

The dual problem is

\[
\begin{align*}
\text{max} \quad W &= 500y_1 + 6y_2 + 10y_3 + 8y_4 \\
\text{s.t.} \\
400y_1 + 3y_2 + 2y_3 + 2y_4 &\leq 50 \quad \text{brownie} \\
200y_1 + 2y_2 + 2y_3 + 4y_4 &\leq 20 \quad \text{ice cream} \\
150y_1 + 4y_3 + y_4 &\leq 30 \quad \text{soda} \\
500y_1 + 4y_3 + 5y_4 &\leq 80 \quad \text{cheesecake} \\
y_1, y_2, y_3, y_4 \geq 0
\end{align*}
\]
We consider a salesperson of “nutrients” who is interested in assuming that each dieter meets daily requirements by purchasing calories, sugar, fat and chocolate as “goods”

The decision is to determine the prices charged

\[ y_i = \text{price per unit of required nutrient to sell to dieters} \]

Objective of the salesperson is to set the prices \[ y_i \] so as to maximize revenues from selling to the dieter the daily ration of required nutrients
Now, the dieter can purchase a brownie for 50¢ and have 400 cal, 3 oz of chocolate, 2 oz of sugar and 2 oz of fat.

The sales price $y_i$ must be set sufficiently low to entice the buyer to get the required nutrients from the brownie:

$$400y_1 + 3y_2 + 2y_3 + 2y_4 \leq 50$$

We derive similar constraints for the ice cream, the soda and the cheesecake.
DUAL PROBLEMS

\[
\begin{align*}
\text{max} & \quad Z = c^T x \\
\text{s.t.} & \quad A x \leq b \\
& \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{min} & \quad W = b^T y \\
\text{s.t.} & \quad A^T y \geq c \\
& \quad y \geq 0
\end{align*}
\]
WEAK DUALITY THEOREM

For any \( x \) feasible for \((P)\) and any \( y \) feasible for \((D)\), the following relation is satisfied

\[ c^T x \leq b^T y \]

Proof:

\[ A^T y \geq c \Rightarrow c^T \leq y^T A \Rightarrow c^T x \leq y^T A x \]

\[ c^T x \leq y^T A x \leq y^T b = b^T y \]
COROLLARY 1 OF THE WEAK DUALITY THEOREM

\[ x \text{ is feasible for } (P) \implies c^T x \leq y^T b \]

for any feasible \( y \) for \( (D) \)

\[ c^T x \leq y^*^T b = \min W \]

for any feasible \( x \) for \( (P) \),

\[ c^T x \leq \min W \]
COROLLARY 2 OF THE WEAK DUALITY THEOREM

\[
y \text{ is feasible for } (D) \Rightarrow c^T x \leq y^T b
\]

for every feasible \( x \) for \((P)\)

\[
\max Z = \max c^T x = c^T x^* \leq y^T b
\]

for any feasible \( y \) of \((D)\),

\[
y^T b \geq \max Z
\]
COROLLARIES 3 AND 4 OF THE WEAK DUALITY THEOREM

If \(( P )\) is feasible and \( \max Z \) is unbounded, i.e.,

\[
Z \rightarrow +\infty ,
\]
then, \(( D )\) has no feasible solution.

If \(( D )\) is feasible and \( \min Z \) is unbounded, i.e.,

\[
Z \rightarrow -\infty ,
\]
then, \(( P )\) is infeasible.
Consider the maximization problem

\[
\text{max } Z = x_1 + 2x_2 + 3x_3 + 4x_4 = \begin{bmatrix} 1, 2, 3, 4 \end{bmatrix} x \\
\text{s.t.}
\]

\[
\begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 1 & 3 & 2 \end{bmatrix} x \leq \begin{bmatrix} 20 \\ 20 \end{bmatrix} \\
\begin{bmatrix} \geq 0 \end{bmatrix}
\]

\((P)\)
The corresponding dual is given by

\[
\begin{align*}
\min & \quad W = b^T y \\
\text{s.t.} & \quad A^T y \geq c \\
& \quad y \geq 0
\end{align*}
\]

With the appropriate substitutions, we obtain
\begin{align*}
\min \quad & W = 20y_1 + 20y_2 \\
\text{s.t.} \quad & y_1 + 2y_2 \geq 1 \\
& 2y_1 + y_2 \geq 2 \\
& 2y_1 + 3y_2 \geq 3 \\
& 3y_1 + 2y_2 \geq 4 \\
& y_1 \geq 0, \quad y_2 \geq 0
\end{align*}
Consider the primal decision

\[ x_i = 1, \quad i = 1, 2, 3, 4; \]

decision is feasible for \((P)\) with

\[ Z = c^T x = 10 \]

The dual decision

\[ y_i = 1, \quad i = 1, 2 \]

is feasible for \((D)\) with

\[ W = b^T y = 40 \]
Clearly,

\[ Z(x_1, x_2, x_3, x_4) = 10 \leq 40 = W(y_1, y_2) \]

and so clearly, the feasible decision for \((P)\) and \((D)\) satisfy the *Weak Duality Theorem*

Moreover, we have

**corollary 1** \(\Rightarrow\) \(10 \leq \min W = W(y_1^*, y_2^*)\)

**corollary 2** \(\Rightarrow\) \(\max Z = Z(x_1^*, x_2^*, x_3^*, x_4^*) \leq b^T y = 40\)
COROLLARIES 5 AND 6

(P) is feasible and (D) is infeasible, then,

(P) is unbounded

☐

(D) is feasible and (P) is infeasible, then,

(D) is unbounded

☐
EXAMPLE

Consider the primal dual problems:

\[
\begin{align*}
\text{max } Z &= x_1 + x_2 \\
\text{s.t.} \\
-x_1 + x_2 + x_3 &\leq 2 \\
-2x_1 + x_2 - x_3 &\leq 1 \\
x_1, x_2, x_3 &\geq 0 \\
\end{align*}
\]

\[
\begin{align*}
\text{min } W &= 2y_1 + y_2 \\
\text{s.t.} \\
-y_1 - 2y_2 &\geq 1 \\
y_1 + y_2 &\geq 1 \\
y_1 - y_2 &\geq 0 \\
y_1, y_2 &\geq 0 \\
\end{align*}
\]

Now

\[x = 0\] is feasible for \((P)\)
EXAMPLE

\[ \mathbf{x} = 0 \text{ is feasible for } (P) \]

but

\[ -y_1 - 2y_2 \geq 1 \]

is impossible for \((D)\) since it is inconsistent with

\[ y_1, y_2 \geq 0 \]

\[ \square \text{ Since } (D) \text{ is infeasible, it follows from Corollary 5} \]

that \( Z \to \infty \)

\[ \square \text{ You are able to show this result by solving } (P) \]

using the simplex scheme
OPTIMALITY CRITERION THEOREM

We consider the primal-dual problems \((P)\) and \((D)\) with

\[
\begin{align*}
\overline{x}^0 & \text{ is feasible for } (P) & \overline{x}^0 & \text{ is optimal for } (P) \\
\overline{y}^0 & \text{ is feasible for } (D) & \Rightarrow & \text{ and } & \overline{y}^0 & \text{ is optimal for } (D) \\
\end{align*}
\]

We next provide the proof:

\[
\begin{align*}
\overline{x}^0 & \text{ is feasible for } (P) & \text{ Weak Duality} & c^T \overline{x}^0 & \leq b^T \overline{y}^0 \\
\overline{y}^0 & \text{ is feasible for } (D) & \Rightarrow & \text{ Theorem} & c^T \overline{x}^0 & \leq b^T \overline{y}^0 \\
\end{align*}
\]
but we are given that
\[ c^T x^0 = b^T y^0 \]
and so it follows that \( \forall \) feasible \( x \) with \( y^0 \) feasible
\[ c^T x \leq b^T y^0 = c^T x^0 \]
and so \( x^0 \) is optimal;

similarly, \( \forall \) feasible \( y \) with \( x^0 \) feasible
\[ b^T y \geq c^T x^0 = b^T y^0 \]
and so it follows that \( y^0 \) is optimal.
(P) is feasible and (D) is feasible; then,

\[ \exists x^* \text{ feasible for } (P) \text{ which is optimal and} \]

\[ \exists y^* \text{ feasible for } (D) \text{ which is optimal such that} \]

\[ c^T x^* = b^T y^* \]
COMPLEMENTARY SLACKNESS CONDITIONS

- \( \bar{x}^* \) and \( \bar{y}^* \) are optimal for \((P)\) and \((D)\) respectively, if and only if

\[
0 = \left( \bar{y}^* A - c^T \right) \bar{x}^* + \bar{y}^T \left( b - A \bar{x}^* \right)
\]

\[
= \bar{y}^* b - c^T \bar{x}^*
\]

- We prove this equivalence result by defining the slack variables \( u \in \mathbb{R}^m \) and \( v \in \mathbb{R}^n \) such that \( \bar{x} \) and \( \bar{y} \) are feasible; at the optimum,

\[
A \bar{x}^* + u^* = b \quad \bar{x}^*, \ u^* \geq 0
\]

\[
A^T \bar{y}^* - v^* = c \quad \bar{y}^*, \ v^* \geq 0
\]
COMPLEMENTARY SLACKNESS CONDITIONS

where the optimal values of the slack variables $u^*$ and $v^*$ depend on the optimal values $x^*$ and $y^*$

Now,

$$y^T A x^* + y^T u^* = y^T b = b^T y^*$$

$$x^T A^T y^* - x^T v^* = x^T c = c^T x^*$$

$$y^T A x^*$$
This implies that

\[ y^*^T u^* + v^*^T x^* = b^T y^* - c^T x^* \]

We need to prove optimality which is true if and only if

\[ y^*^T u^* + v^*^T x^* = 0 \]
However, \( x^*, y^* \) are optimal

\[
\begin{align*}
\mathbf{c}^T x^* &= \mathbf{b}^T y^* \\
\Rightarrow \quad y^T \mathbf{u}^* + v^T x^* &= 0
\end{align*}
\]

Also, \( y^T \mathbf{u}^* + v^T x^* = 0 \Rightarrow \mathbf{b}^T y^* = \mathbf{c}^T x^* \)

\( x^* \) is optimal for (P) and \( y^* \) is optimal for (D)
COMPLEMENTARY SLACKNESS CONDITIONS

Note that

\[ x^*, y^*, u^*, v^* > 0 \Rightarrow \text{component-wise each element} \geq 0 \]

\[ y^* u^* + v^* x^* = 0 \Rightarrow y_i^* u_i^* = 0 \quad \forall i = 1, \ldots, m \]

and \[ v_j^* x_j^* = 0 \quad \forall j = 1, \ldots, n \]

At the optimum,

\[ y_i^* \left( b_i - \sum_{j=1}^{n} a_{ij} x_j^* \right) = 0 \quad i = 1, \ldots, m \]

and

\[ x_j^* \left( \sum_{i=1}^{m} a_{ji} y_i^* - c_j \right) = 0 \quad j = 1, \ldots, n \]
COMPLEMENTARY SLACKNESS CONDITIONS

- Hence, for \( i = 1, 2, \ldots, m \)

  \[
y_i^* > 0 \implies b_i = \sum_{j=1}^{n} a_{ij} x_j^*
\]

  and

  \[
b_i - \sum_{j=1}^{m} a_{ij} x_i^* > 0 \implies y_i^* = 0
\]

- Similarly for \( j = 1, 2, \ldots, n \)

  \[
x_i^* > 0 \implies \sum_{i=1}^{m} a_{ji} y_i^* = c_j
\]

  and

  \[
\sum_{i=1}^{m} a_{ji} y_i^* - c_j > 0 \implies x_j^* = 0
\]
EXAMPLE

\[ \text{max} \quad Z = x_1 + 2x_2 + 3x_3 + 4x_4 \]

s.t.

\[ x_1 + 2x_2 + 2x_3 + 3x_4 \leq 20 \]  \hspace{1cm} (P)

\[ 2x_1 + x_2 + 3x_3 + 2x_4 \leq 20 \]

\[ x_i \geq 0 \quad i = 1, \ldots, 4 \]
EXAMPLE

\[ \text{min} \quad W = 20y_1 + 20y_2 \]

\[ \text{s.t.} \]

\[ y_1 + 2y_2 \geq 1 \]

\[ 2y_1 + y_2 \geq 2 \]

\[ 2y_1 + 3y_2 \geq 3 \]

\[ 3y_1 + 2y_2 \geq 4 \]

\[ y_1, y_2 \geq 0 \]
EXAMPLE

\[
x^*, y^* \text{ optimal } \Rightarrow
\]

\[
y_1^* \left(20 - x_1^* - 2x_2^* - 2x_3^* - 3x_4^*\right) = 0
\]

\[
y_2^* \left(20 - 2x_1^* - x_2^* - 3x_3^* - 2x_4^*\right) = 0
\]

\[
y^* = \begin{bmatrix} 1.2 \\ 0.2 \end{bmatrix}
\]

is given as an optimal solution with

\[
\begin{align*}
\text{min } W &= 28
\end{align*}
\]
**EXAMPLE**

\[
x_1^* + 2x_2^* + 2x_3^* + 3x_4^* = 20
\]

\[
2x_1^* + x_2^* + 3x_3^* + 2x_4^* = 20
\]

\[
y_1^* + 2y_2^* = 1.2 + 0.4 > 1 \implies x_1^* = 0
\]

\[
2y_1^* + y_2^* = 2.4 + 0.2 > 2 \implies x_2^* = 0
\]

\[
2y_1^* + 3y_2^* = 2.4 + 0.6 = 3
\]

\[
3y_1^* + 2y_2^* = 3.6 + 0.4 = 4
\]

so that

\[
2x_3^* + 3x_4^* = 20 \implies x_3^* = 4
\]

\[
3x_3^* + 2x_4^* = 20 \implies x_4^* = 4
\]
Key uses of *c.s. conditions* are

- finding optimal \((P)\) solution given optimal \((D)\) solution and vice versa
- verification of optimality of solution (whether a feasible solution is optimal)

We can start with a feasible solution and attempt to construct an optimal dual solution; if we succeed, then the feasible primal solution is *optimal*.
DUALITY

$max \quad Z = c^T x$

$s.t. \quad Ax \leq b$

$s.t. \quad x \geq 0$

$min \quad W = b^T y$

$s.t. \quad A^T y \geq c$

$s.t. \quad y \geq 0$

$(P)$

$(D)$
Suppose the primal problem is minimization, then,

\[
\begin{align*}
\text{min} & \quad Z = c^T x \\
\text{s.t.} & \quad Ax \geq b \\
& \quad x \geq 0
\end{align*}
\]  

\[
\begin{align*}
\text{max} & \quad W = b^T y \\
\text{s.t.} & \quad A^T y \leq c \\
& \quad y \geq 0
\end{align*}
\]

\( (P) \)  

\( (D) \)
The economic interpretation is

\[ Z^* = \max Z = c^T x^* = b^T y^* = W^* = \min W \]

\[ b_i - \text{constrained resource quantities} \quad i = 1, 2, \ldots, m \]

\[ y_i^* - \text{optimal dual variables} \]

Suppose, we change

\[ b_i \rightarrow b_i + \Delta b_i \Rightarrow \Delta Z = y_i^* \Delta b_i \]

In words, the optimal dual variable for each primal constraint gives the net change in the optimal value of the objective function \( Z \) for a one unit change in the constraint on resources.
Economists refer to the dual variable as the *shadow price* on the constraint resource.

The *shadow price* determines the value/worth of having an additional quantity of a resource.

In the previous example, the optimal dual variables indicate that the worth of another unit of resource 1 is 1.2 while that of another unit of resource 2 is 0.2.
We start out with

\[
\begin{align*}
\text{max} & \quad Z = \epsilon^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]
To find $(D)$, we first put $(P)$ in symmetric form

\[
\begin{align*}
\begin{array}{c}
y_+ 
\end{array} & \iff 
\begin{array}{c}
Ax 
\end{array} \leq 
\begin{array}{c}
b 
\end{array} 
\begin{bmatrix}
A \\
A 
\end{bmatrix} 
\begin{array}{c}
x 
\end{array} \leq 
\begin{bmatrix}
b \\
b 
\end{bmatrix} 
\text{symmetric form} \\
\begin{array}{c}
y_- 
\end{array} & \iff 
\begin{array}{c}
-Ax 
\end{array} \leq 
\begin{array}{c}
-b 
\end{array} 
\begin{bmatrix}
-A \\
-A 
\end{bmatrix} 
\begin{array}{c}
x 
\end{array} \leq 
\begin{bmatrix}
-b \\
-b 
\end{bmatrix} \\
\begin{array}{c}
x 
\end{array} \geq 
\begin{array}{c}
0 
\end{array} 
\end{align*}
\]
Let
\[ y = y_+ - y_- \]

We rewrite the problem as
\[
\begin{align*}
\min W &= b^T y \\
\text{s.t.} \quad A^T y &\geq c \\
\quad y \text{ is unsigned}
\end{align*}
\]

The c.s. conditions apply
\[
\begin{align*}
x^*^T \left( A^T y^* - c \right) &= 0
\end{align*}
\]
EXAMPLE 5: THE PRIMAL

\[ \text{max } Z = x_1 - x_2 + x_3 - x_4 \]

\[ \text{s.t.} \]

\[ y_1 \leftrightarrow x_1 + x_2 + x_3 + x_4 = 8 \]

\[ y_2 \leftrightarrow x_1 \leq 8 \]

\[ y_3 \leftrightarrow x_2 \leq 4 \]

\[ y_4 \leftrightarrow -x_2 \leq 4 \]

\[ y_5 \leftrightarrow x_3 \leq 4 \]

\[ y_6 \leftrightarrow -x_3 \leq 2 \]

\[ y_7 \leftrightarrow x_4 \leq 10 \]

\[ x_1, x_4 \geq 0 \]

\[ x_2, x_3 \text{ unsigned} \]
EXAMPLE 5: THE DUAL

\[ \begin{align*}
\min W &= 8y_1 + 8y_2 + 4y_3 + 4y_4 + 4y_5 + 2y_6 + 10y_7 \\
\text{s.t.} & \\
\quad x_1 & \leftrightarrow y_1 + y_2 \quad \geq 1 \\
x_2 & \leftrightarrow y_1 + y_3 - y_4 \quad = -1 \quad (D) \\
x_3 & \leftrightarrow y_1 \quad + y_5 - y_6 \quad = 1 \\
x_4 & \leftrightarrow y_1 \quad + y_7 \geq -1 \\
y_2, \ldots, y_7 & \geq 0 \\
y_1 & \text{ unsigned} 
\end{align*} \]
EXAMPLE 5: c.s. conditions

We are given that

\[ x^* = \begin{bmatrix} 8 \\ -4 \\ 4 \\ 4 \\ 0 \end{bmatrix} \]

is optimal for \((P)\)

Then the c.s. conditions obtain

\[ x^*_1 \left( y^*_1 + y^*_2 - 1 \right) = 0 \]
EXAMPLE 5: c.s. conditions

so that

\[ x_1^* = 8 > 0 \Rightarrow y_1^* + y_2^* = 1 \]

and so \( y_2^* = 1 - y_1^* \)

- The other c.s. conditions require

\[ y_i^* \left( \sum_{j=1}^{4} a_{ij} x_j^* - b_i \right) = 0 \]

- Now, \( x_4^* = 0 \) implies \( x_4^* - 10 < 0 \) and so \( y_7^* = 0 \)
EXAMPLE 5: c.s. conditions

Also, $x^*_3 = 4$ implies

$$y^*_6 = 0$$

We similarly use the c.s. conditions

$$x^*_j \left( \sum_{i=1}^{7} a_{ji} y^*_i - c_j \right) = 0$$

to provide implications on the $y^*_i$ variables
EXAMPLE 5: c.s. conditions

- Since \( x^*_2 = -4 \), then we have
  \[
y^*_3 = 0
  \]

- Now, with \( y^*_7 = 0 \) we have
  \[
y^*_1 > -1
  \]

- Now, we have already shown that
  \[
y^*_2 = 1 - y^*_1
  \]
EXAMPLE 5

Suppose that

\[ y_1^* = 1 \]

and so,

\[ y_2^* = 0 \]

Furthermore,

\[ y_1^* + y_3^* - y_4^* = 1 - y_4^* = -1 \]

implies that

\[ y_4^* = 2 \]
Also

\[ y_1^* + y_5^* - y_6^* = 1 \]

implies

\[ 1 + y_5^* = 1 \]

and so

\[ y_5^* = 0 \]
EXAMPLE 5

Therefore, as \( \mathbf{W} = \mathbf{b}^T \mathbf{y} \)

\[
\mathbf{W}(\mathbf{y}^*) = (8)(1) + (8)(0) + (4)(0) + (4)(2) + \\
(4)(0) + (2)(0) + (10)(0)
\]

\[= 16\]

and so

\[\mathbf{W}^* = 16 = \mathbf{Z}^* \iff \text{optimality of } (P) \text{ and } (D)\]
### PRIMAL – DUAL TABLE

<table>
<thead>
<tr>
<th><strong>primal (maximize)</strong></th>
<th><strong>dual (minimize)</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>$A$ (coefficient matrix)</strong></td>
<td><strong>$A^T$ (transpose of the coefficient matrix)</strong></td>
</tr>
<tr>
<td><strong>$b$ (right-hand side vector)</strong></td>
<td><strong>$b$ (cost vector)</strong></td>
</tr>
<tr>
<td><strong>$c$ (price vector)</strong></td>
<td><strong>$c$ (right hand side vector)</strong></td>
</tr>
<tr>
<td>$i^{th}$ constraint is $=$ type</td>
<td>the dual variable $y_i$ is unrestricted in sign</td>
</tr>
<tr>
<td>$i^{th}$ constraint is $\leq$ type</td>
<td>the dual variable $y_i \geq 0$</td>
</tr>
<tr>
<td>$i^{th}$ constraint is $\geq$ type</td>
<td>the dual variable $y_i \leq 0$</td>
</tr>
<tr>
<td>$x_j$ is unrestricted</td>
<td>$j^{th}$ dual constraint is $=$ type</td>
</tr>
<tr>
<td>$x_j \geq 0$</td>
<td>$j^{th}$ dual constraint is $\geq$ type</td>
</tr>
<tr>
<td>$x_j \leq 0$</td>
<td>$j^{th}$ dual constraint is $\leq$ type</td>
</tr>
</tbody>
</table>