ECE 307 – Techniques for Engineering Decisions

3. Introduction to the Simplex Algorithm

George Gross

Department of Electrical and Computer Engineering

University of Illinois at Urbana-Champaign

SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

- ☐ We examine the solution of

$$\underline{A}\,\underline{x} = \underline{b}$$

using *Gauss−Jordan* elimination

- ☐ We first use a simple example and then generalize to cases of general interest
- ☐ Consider the system of two equations in five unknowns:

SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

$$S_1 \begin{cases} x_1 - 2x_2 + x_3 - 4x_4 + 2x_5 = 2 & (i) \\ x_1 - x_2 - x_3 - 3x_4 - x_5 = 4 & (ii) \end{cases}$$

- ☐ **For this simple example, the number of unknowns exceeds the number of equations and so the system has multiple solutions; this is the principal reason that the *LP* solution is *nontrivial***

SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

- **The** $Gauss — Jordan$ **elimination uses** *elementary row operations*:
  - **multiplication of any equation by a nonzero constant**
  - **addition to any equation of a nonzero constant multiple of any other equation**

SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

- **We transform system S_1 by multiplication of equation (i) by -1 and its addition to equation (ii) so as to zero out the coefficient of x_1 to obtain**

$$S_2 \begin{cases} x_1 - 2x_2 + x_3 - 4x_4 + 2x_5 = 2 \\ \qquad x_2 - 2x_3 + x_4 - 3x_5 = 2 \end{cases}$$

DEFINITIONS

- A ***basic variable*** **is a variable x_i that appears with the coefficient 1 in an equation and with the coefficient *0* in all the other equations**
- **A variable x_j that is *not* basic is called a *nonbasic variable***
- **In the system S_2, x_1 appears as a *basic* variable; x_2, x_3, x_4 and x_5 are *nonbasic* variables**
- **Basic variables may be generated through the use of *elementary row operations***

DEFINITIONS

- ☐ A *pivot operation* is the set of sequential elementary row operations that reduces a system of linear equations into the form in which a specified variable becomes a *basic variable*
- ☐ A *canonical system* is a set of linear equations obtained through *pivot operations* with the property that the system has the same number of *basic variables* as the number of equations in the set

CANONICAL SYSTEM FORM

- ☐ We transform the system S_2 into the canonical form of system S_3 :

$$S_3 \begin{cases} x_1 \qquad\quad - 3x_3 - 2x_4 - 4x_5 = 6 \\ \qquad x_2 - 2x_3 + x_4 - 3x_5 = 2 \end{cases}$$

- ☐ The ***basic solution*** is obtained from a canonical system with **all the nonbasic variables set to *0***
- ☐ For the example, we set $x_3 = x_4 = x_5 = 0$ and so

$$x_1 = 6 \text{ and } x_2 = 2$$

BASIC FEASIBLE SOLUTION

- ☐ A *basic feasible solution* is a basic solution in which the value of each *basic variable* is nonnegative
- ☐ In the example of system S_2, we may choose any two variables to be basic
- ☐ In general for a system of m equations in n unknowns there are $\begin{pmatrix} n \\ m \end{pmatrix}$ possible combinations of basic variables

BASIC FEASIBLE SOLUTION

- ☐ As *n* increases, the number of combinations

  becomes large, even though it remains finite

- ☐ For the example, we have

$$\binom{5}{2} = \frac{5!}{3!\,2!} = 10$$

  combinations of possible choices

THE SIMPLEX SOLUTION METHOD

- ☐ We next use a simple example to construct the *simplex* solution method
- ☐ The *simplex method* is a *systematic and efficient scheme* to examine a *subset* of the basic feasible solutions of the *LP* to hone in on *an optimal solution*
- ☐ We apply the notions introduced in the definitions we introduced above

SIMPLEX METHODOLOGY EXAMPLE

$$\max Z = 5x_1 + 2x_2 + 3x_3 - x_4 + x_5$$

$$s.t.$$

$$\text{canonical form} \left\{ \begin{array}{lcr} x_1 + 2x_2 + 2x_3 + x_4 & = 8 & (*) \\ 3x_1 + 4x_2 + x_3 + x_5 & = 7 & (**) \end{array} \right.$$

$$x_i \geq 0 \qquad i = 1, \ldots, 5$$

THE SIMPLEX SOLUTION METHOD

- ☐ The *canonical form* of the example allows the determination of a basic feasible solution

$$x_1 = x_2 = x_3 = 0 \qquad x_4 = 8, \;\; x_5 = 7$$

- ☐ The corresponding value of the objective is

$$Z = -8 + 7 = -1$$

- ☐ The next step is to improve the *basic feasible solution* and we need to find an *adjacent basic feasible solution*

ADJACENT FEASIBLE SOLUTION

- An *adjacent basic feasible solution* is one which differs from the current basic feasible solution in *exactly one basic variable*

- Note, we characterize a *basic feasible solution* by the following traits

$$\textit{basic variable} \geq 0$$

$$\textit{nonbasic variable} = 0$$

ADJACENT FEASIBLE SOLUTION

- ☐ The search for an adjacent basic feasible solution is based on the idea of the switch of a *nonbasic* variable into a *basic* variable by increasing its value from *0* to the largest positive value without the *violation* of any constraints
- ☐ To make the search efficient, we select *the nonbasic* variable that improves the value of Z by the largest amount for the *maximization objective*

ADJACENT FEASIBLE SOLUTION

- ☐ **In the example, consider the *nonbasic* variable x_1, we leave $x_2 = x_3 = 0$ and examine the possibility to convert x_1 into a basic variable**
- ☐ **The variable x_1 enters in both constraints**

$$\begin{aligned} x_1 + x_4 \quad &= 8 \\ 3x_1 + \quad x_5 &= 7 \end{aligned}$$

ADJACENT FEASIBLE SOLUTION

- ☐ **The largest value x_1 may assume without making either x_4 or x_5 negative is**

$$min \left\{ 8,\ \frac{7}{3} \right\} = \frac{7}{3}$$

- ☐ **We have the new *basic* variable with the value**

$$x_1 = \frac{7}{3} \quad ,$$

and the other *basic* variable is

$$x_4 = \frac{17}{3}$$

ADJACENT FEASIBLE SOLUTION

and the three *nonbasic* variables are set to 0:

$$x_2 = x_3 = 0 \text{ and } x_5 = 0$$

- ☐ **Note that we obtain an improvement in Z since its value becomes**

$$Z = 5 \cdot \frac{7}{3} - \frac{17}{3} = \frac{18}{3} = 6 > -1$$

- ☐ **We next transform the system of equations into *canonical form*:**

SIMPLEX METHODOLOGY EXAMPLE

$$\max Z = 5x_1 + 2x_2 + 3x_3 - x_4 + x_5$$

$s.t.$

non-canonical form for x_1

$$\left\{ \begin{array}{ll} x_1 + 2x_2 + 2x_3 + x_4 = 8 & (*) \\ 3x_1 + 4x_2 + x_3 + x_5 = 7 & (**) \end{array} \right.$$

$$x_i \geq 0 \qquad i = 1, \ldots, 5$$

ADJACENT FEASIBLE SOLUTION

- ○ multiply equation $(**)$ by $-\frac{1}{3}$ and add to equation $(*)$

$$\frac{2}{3}x_2 + \frac{5}{3}x_3 + x_4 - \frac{1}{3}x_5 = \frac{17}{3}$$

- ○ multiply equation $(**)$ by $\frac{1}{3}$

$$x_1 + \frac{4}{3}x_2 + \frac{1}{3}x_3 + \frac{1}{3}x_5 = \frac{7}{3}$$

THE SIMPLEX SOLUTION METHOD

- We continue this process until the *condition of optimality* is satisfied:
  - in a maximization problem, a *basic feasible solution* is *optimal* if and only if the relative profits of each *nonbasic variable* is ≤ *0*
  - in a minimization problem, a basic feasible solution is optimal if and only if the relative costs of each *nonbasic variable* is ≥ *0*

THE SIMPLEX SOLUTION METHOD

- **The *relative profits (costs)* are given by the change in *Z* corresponding to a unit change in a *nonbasic variable***
- **We use this fact to select the next *nonbasic variable* to enter the basis**

SIMPLEX ALGORITHM FOR MAXIMIZATION

Step 1: start with an *initial basic feasible solution* with all constraint equations in *canonical form*

Step 2: check for optimality condition: if the relative profits are ≤ 0 for each *nonbasic variable*, then the *basic feasible solution* is optimal and *stop*; else, go to Step 3

SIMPLEX ALGORITHM FOR MAXIMIZATION

Step 3: select a *nonbasic variable* to become the new *basic variable*; check the limits on the *nonbasic variable* – the limiting constraint determines which *basic variable* is replaced by the selected *nonbasic variable*

Step 4: construct the *canonical form* for the new set of basic variables through *elementary row operations*; evaluate the *basic feasible solution* and Z and return to Step 2

THE SIMPLEX TABLEAU

- ☐ We use an efficient way to visually represent the steps in the simplex method through a sequence of so–called *tableaus*
- ☐ We illustrate the tableau for the simple example for the initial basic feasible solution

THE SIMPLEX TABLEAU

coefficients of the basic variables in Z

coefficient of x_j in Z

$\underline{c}_B$	c_j	5	2	3	-1	1	*constraint constants*
	basic variables	x_1	x_2	x_3	x_4	x_5	
-1	x_4	1	2	2	1		8
1	x_5	3	4	1		1	7

THE SIMPLEX TABLEAU

- ☐ **The optimality check requires the evaluation of**

$$\tilde{c}_j = c_j - \left(\underline{c}_B^T \cdot \begin{matrix} \textit{column corresponding} \\ \textit{to } x_j \textit{ in canonical form} \end{matrix} \right)$$

- ☐ **For each *nonbasic variable* x_j, for our example, we have**

$$\tilde{c}_1 = 5 - [-1, 1] \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3$$

$$\tilde{c}_2 = 2 - [-1, 1] \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \mathit{0}$$

$$\tilde{c}_3 = 3 - [-1, 1] \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4$$

THE SIMPLEX TABLEAU

- ❑ We interpret each $\tilde{c}_j$ as the change in Z in response to a unit increase in x_j

$\underline{c}_B$	c_j	5	2	3	-1	1	*constraint*
	basic variables	x_1	x_2	x_3	x_4	x_5	*constants*
-1	x_4	1	2	2	1		8
1	x_5	3	4	1		1	7
$\underline{\tilde{c}}^T$		3	*0*	4	*0*	*0*	$Z = -1$

SIMPLEX TABLEAU

- ☐ **Note that the optimality test indicates that**

  $$\tilde{c}_1 = 3 > 0 \quad \text{and} \quad \tilde{c}_3 = 4 > 0$$

  and so the *initial basic feasible solution* is not *optimal*

- ☐ **Since $\tilde{c}_3 > \tilde{c}_1$, we pick x_3 as the *nonbasic variable* to enter as a *basic variable***

- ☐ **We examine the limiting solution for x_3 in the two constraint equations:**

THE SIMPLEX TABLEAU

equation	*limiting basic variable*	*upper limit on* x_3
1	x_4	(8/2) = 4
2	x_5	(7/1) = 7

and so the limiting value is

$$min\ \{4,7\} = 4$$

- ❑ **We replace the basic variable** x_4 **by** x_3

SIMPLEX METHODOLOGY EXAMPLE

$$max\ Z = 5x_1 + 2x_2 + 3x_3 - x_4 + x_5$$

$s.t.$

canonical form in x_4 *and* x_5

$$\begin{cases} x_1 + 2x_2 + 2x_3 + x_4 = 8 & (*) \\ 3x_1 + 4x_2 + x_3 + x_5 = 7 & (**) \end{cases}$$

$$x_i \geq 0 \qquad i = 1, \ldots, 5$$

THE SIMPLEX TABLEAU

- For the new basic feasible solution, we put the equations into canonical form by
  - multiplication of $(*)$ by $\frac{1}{2}$ to obtain $(*\dagger)$
  - subtraction of $(*\dagger)$ from $(**)$ to obtain $(**\dagger)$

$$
\begin{array}{lllllllll}
\frac{1}{2}x_1 & + & x_2 & + & x_3 & + & \frac{1}{2}x_4 & & & = & 4 & (*\dagger) \\
\frac{5}{2}x_1 & + & 3x_2 & & & - & \frac{1}{2}x_4 & + & x_5 & = & 3 & (**\dagger)
\end{array}
$$

- The adjacent basic feasible solution is

$$x_1 = x_2 = x_4 = 0 \qquad x_3 = 4, \quad x_5 = 3$$

THE SIMPLEX TABLEAU

$\underline{c}_B$	c_j *basic variables*	5	2	3	– 1	1	*constraint constants*
		x_1	x_2	x_3	x_4	x_5	
3	x_3	1/2	1	1	1/2		4
1	x_5	5/2	3		–1/2	1	3
$\tilde{\underline{c}}^T$		1	– 4	*0*	– 2	*0*	$\underbrace{Z = 15}_{x_3 = 4,\ x_5 = 3}$

THE SIMPLEX TABLEAU

- **Since $\tilde{c}_1 > 0$, the basic feasible solution is non-optimal**

- **We examine how to bring x_1 into the basis**

equation	*limiting basic variable*	*upper limit on x_1*
(*†)	x_3	4/(1/2) = 8
(**†)	x_5	3/(5/2) = 6/5

THE SIMPLEX TABLEAU

- The variable x_1 enters the basis with the value

$$min\left\{ 8, \ \frac{6}{5} \right\} = \frac{6}{5}$$

and x_5 is replaced as a basic variable by x_1

- We need to put the equations

$$\begin{array}{lllllllll} \frac{1}{2}x_1 & + & x_2 & + & x_3 & + & \frac{1}{2}x_4 & & & = & 4 & (*\dagger) \\ \frac{5}{2}x_1 & + & 3x_2 & & & - & \frac{1}{2}x_4 & + & x_5 & = & 3 & (**\dagger) \end{array}$$

into *canonical form* for the *basic variables* x_3 and x_1

THE SIMPLEX TABLEAU

- The following elementary row operations are used
  - multiply (* * †) by – 1/5 and add to (* †)

$$\frac{2}{5}x_2 + x_3 + \frac{3}{5}x_4 - \frac{1}{5}x_5 = \frac{17}{5}$$

  - multiply (* * †) by 2/5

$$x_1 + \frac{6}{5}x_2 - \frac{1}{5}x_4 + \frac{2}{5}x_5 = \frac{6}{5}$$

and construct the corresponding tableau

THE SIMPLEX TABLEAU

$\underline{c}_B$	c_j	5	2	3	-1	1	*constraint constants*
	basic variables	x_1	x_2	x_3	x_4	x_5	
3	x_3		2/5	1	3/5	– 1/5	17/5
5	x_1	1	6/5		– 1/5	2/5	6/5
$\underline{\tilde{c}}^T$		*0*	– 26/5	*0*	– 9/5	– 2/5	$Z = 81/5$

$\tilde{c}_j \leq 0$ *implies optimality*

$\overbrace{16.2 > 15}$

SIMPLEX TABLEAU EXAMPLE

$$max\ Z = 3x_1 + 2x_2$$

$$s.t.$$

$$-x_1 + 2x_2 \leq 4$$

$$3x_1 + 2x_2 \leq 14$$

$$x_1 - x_2 \leq 3$$

$$x_1 \geq 0 \qquad x_2 \geq 0$$

SIMPLEX TABLEAU EXAMPLE

- We put this problem into standard form:

$$\max Z = 3x_1 + 2x_2$$

canonical form

$$s.t.$$

$$\left.\begin{array}{rcrcrcrcrcr} -x_1 & + & 2x_2 & + & x_3 & & & & & = & 4 \\ 3x_1 & + & 2x_2 & & & + & x_4 & & & = & 14 \\ x_1 & - & x_2 & & & & & + & x_5 & = & 3 \end{array}\right\}$$

$$x_1, \dots, x_5 \geq 0$$

- x_3, x_4, x_5 are *fictitious* – or *slack* – variables

SIMPLEX TABLEAU EXAMPLE

$\underline{c}_B$	c_j	3	2	0	0	0	constraint constants
	basic variables	x_1	x_2	x_3	x_4	x_5	
0	x_3	-1	2	1			4
0	x_4	3	2		1		14
0	x_5	1	-1			1	3
$\underline{\tilde{c}}^T$		$\boxed{3}$	2	0	0	0	$Z = 0$

$$\tilde{c}_j = c_j - \left(\underline{c}_B^T \bullet \text{column corresponding to } x_j \right)$$

SIMPLEX TABLEAU EXAMPLE

- ☐ The data in $\underline{\tilde{c}}^{T}$ indicate that the highest relative profits correspond to x_1 *and* so we wish to make x_1 a basic variable
- ☐ To bring x_1 into the basis requires to evaluate

$$min \left\{ \infty, \frac{14}{3}, 3 \right\} = 3$$

and so x_1 replaces x_5 with the value 3

- ☐ We evaluate the basic variable at the adjacent basic feasible solution and convert into canonical form; the new tableau becomes

SIMPLEX TABLEAU EXAMPLE

$\underline{c}_B$	c_j basic variables	3	2	0	0	0	constraint constants
		x_1	x_2	x_3	x_4	x_5	
0	x_3		1	1		1	7
0	x_4		5		1	– 3	5
3	x_1	1	– 1			1	3
$\underline{\tilde{c}}^T$		0	5	0	0	– 3	$Z = 9$

SIMPLEX TABLEAU EXAMPLE

- ☐ We reproduce here the calculation of the $\underline{\tilde{c}}^{T}$ components

$$\tilde{c}_{j} = c_{j} - \left(\underline{c}_{B}^{T} \bullet \textit{column corresponding to } x_{j} \right)$$

for each nonbasic variable x_j

- ☐ Note that, by definition, $\tilde{c}_{i} = 0$ for each basic variable x_i

SIMPLEX TABLEAU EXAMPLE

- ☐ **The calculations give**

$\tilde{c}_1 = 0$ **by definition since** x_1 **is in the basis**

$$\tilde{c}_2 = 2 - \begin{bmatrix} 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix} = 5$$

indicates possible improvement

$\tilde{c}_3 = 0$ **by definition since** x_3 **is in the basis**

$\tilde{c}_4 = 0$ **by definition since** x_4 **is in the basis**

$$\tilde{c}_5 = 0 - \begin{bmatrix} 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} = -3$$

SIMPLEX TABLEAU EXAMPLE

- ☐ Clearly, the only choice is to get x_2 into the basis and so we need to establish the limiting condition from the three equations by evaluating

$$min\left\{7, 1, \infty\right\} = 1$$

and so x_2 replaces x_4, which becomes a nonbasic variable

- ☐ We need to rewrite the equations into canonical form for x_3 and x_2 and construct the new tableau

SIMPLEX TABLEAU EXAMPLE

$\underline{c}_B$	c_j	3	2	0	0	0	constraint constants
	basic variables	x_1	x_2	x_3	x_4	x_5	
0	x_3			1	–1/5	8/5	6
2	x_2		1		1/5	–3/5	1
3	x_1	1			1/5	2/5	4
$\underline{\tilde{c}}^T$		0	0	0	– 1	0	$Z = 14$

$\tilde{c}_j \le 0 \ \forall \ j \quad \Rightarrow$ optimum

SIMPLEX TABLEAU EXAMPLE

- An optimum is at the solution of

$$\left.\begin{array}{rrrrrrl} & & x_3 & - \frac{1}{5}x_4 & + \frac{8}{5}x_5 & = & 6 \\ & x_2 & & + \frac{1}{5}x_4 & - \frac{2}{5}x_5 & = & 1 \\ x_1 & & & + \frac{1}{5}x_4 & + \frac{2}{5}x_5 & = & 4 \end{array}\right\} \textit{canonical form in } x_1, x_2 \textit{ and } x_3$$

SIMPLEX TABLEAU EXAMPLE

- ☐ **This optimum is given by**

$$x_4 = x_5 = 0$$

$$x_3 = 6$$

$$x_2 = 1$$

$$x_1 = 4$$

LINEAR PROGRAMMING EXAMPLE

- ☐ **Consider the following *LP***

$$\max Z = 3x_1 + 2x_2$$

$$s.t.$$

$$-x_1 + 2x_2 \leq 4$$

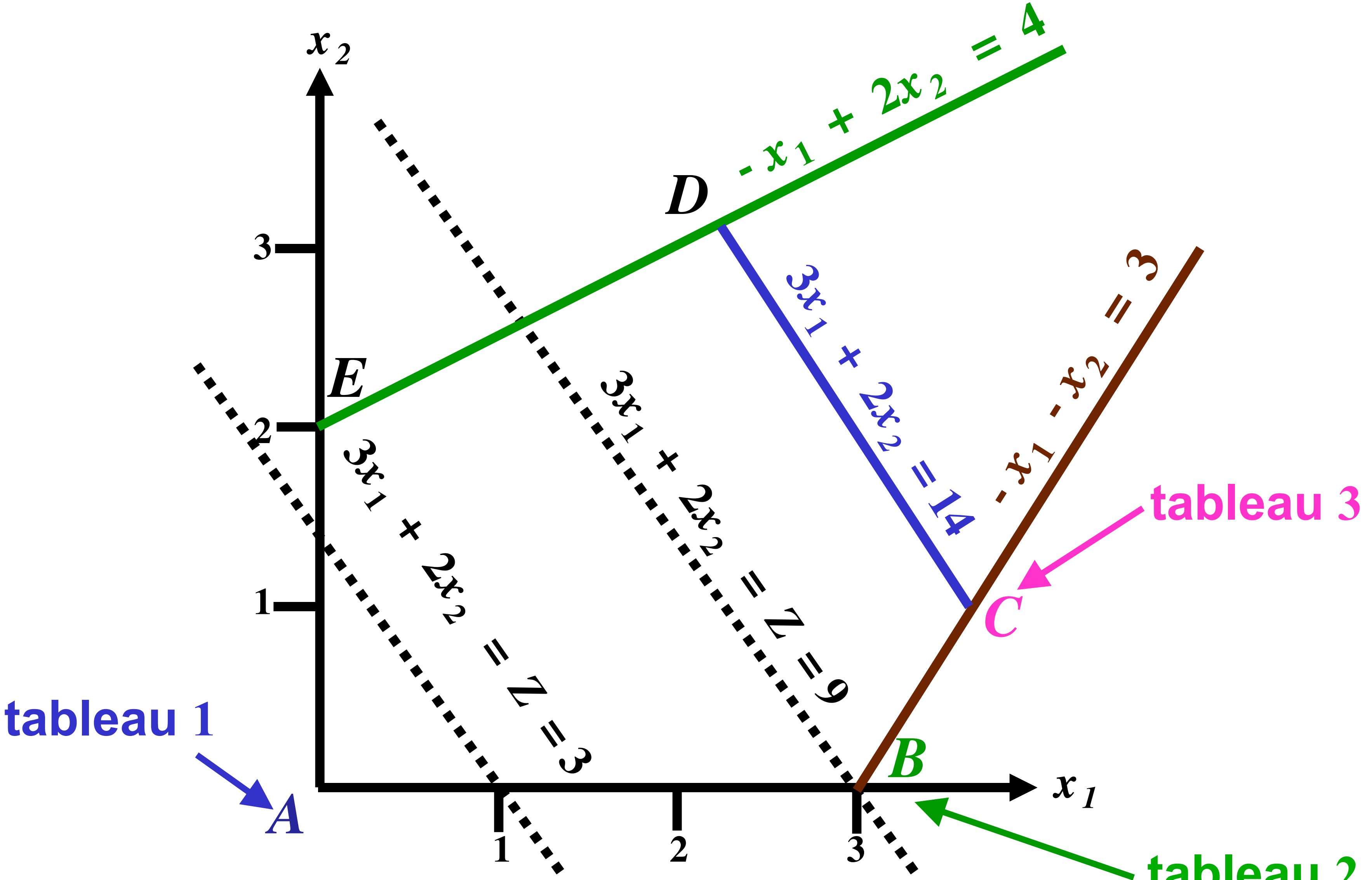
$$3x_1 + 2x_2 \leq 14$$

$$x_1 - x_2 \leq 3$$

$$x_1 \geq 0, \quad x_2 \geq 0$$

- ☐ **The graphical representation corresponds to**

LINEAR PROGRAMMING EXAMPLE



LINEAR PROGRAMMING EXAMPLE

- ☐ **The tableau approach leads to C which is an optimal solution with**

$$x_1 = 4, \; x_2 = 1, \; x_3 = 6, \; x_4 = 0, \; x_5 = 0$$

- ☐ **Note that any point along CD has $Z = 14$ and as such D is another optimal solution corresponding to an adjacent basic feasible solution**
- ☐ **We may obtain D from C by bringing into the basis the nonbasic variable x_5 in Tableau 3; note that $\tilde{c}_5 = 0$**

LINEAR PROGRAMMING EXAMPLE

- **We may choose x_5 as a basic variable without affecting Z since its relative profits are 0 ; we compute the limiting value of x_5**
- **The limit is imposed by x_3 which, consequently, leaves the basis**
- **The corresponding tableau is:**

LINEAR PROGRAMMING EXAMPLE

$\underline{c}_B$	c_j basic variables	3	2	*0*	*0*	*0*	*constraint constants*
		x_1	x_2	x_3	x_4	x_5	
0	x_3			5/8	–1/8	1	15/4
2	x_2		1	3/8	1/8		13/4
3	x_1	1		–1/4	1/4		5/2
$\underline{\tilde{c}}^T$		*0*	*0*	*0*	– 1	*0*	$Z = 14$

$$\tilde{c}_j \leq 0 \quad \forall\ j$$

LINEAR PROGRAMMING EXAMPLE

- ☐ **The adjacent feasible solution is given by**

$$x_1 = \frac{5}{2}, \quad x_2 = \frac{13}{4}, \quad x_3 = x_4 = 0, \quad x_5 = \frac{15}{4}$$

- ☐ **Note that at this basic feasible solution,**

$$\tilde{c}_j \leq 0 \quad \forall j$$

and so this is also an *optimal solution*

ALTERNATE OPTIMAL SOLUTION

In general, an *alternate optimal solution* is indicated whenever there exists a *nonbasic variable* x_j with $\tilde{c}_j = 0$ in an optimal tableau; such a situation indicates a *non unique optimum* for the *LP*

MINIMIZATION LP

- ☐ **Consider a minimization LP with the form given by**

$$\min \qquad Z = \sum_{i=1}^{n} c_i \, x_i$$

$$s.t.$$

$$\underline{A}\underline{x} = \underline{b}$$

$$\underline{x} \geq \underline{0}$$

MINIMIZATION *LP*

- We replace the optimality check in the simplex scheme by the *minimization optimality check*:

  if each coefficient $\tilde{c}_j \geq 0$, stop; else, select the *nonbasic variable* with the *most negative–valued* $\underline{\tilde{c}}$ component to become the *new basic variable*

MINIMIZATION *LP*

- ☐ Every minimization *LP* may be solved as a maximization *LP* because of equivalence

$$\begin{array}{llll} min & Z = \underline{c}^T \underline{x} & max & Z' = (-\underline{c}^T)\underline{x} \\ s.t. & & s.t. & \\ & \underline{A}\,\underline{x} = \underline{b} & & \underline{A}\,\underline{x} = \underline{b} \\ & \underline{x} \geq \underline{0} & & \underline{x} \geq \underline{0} \end{array}$$

with the solutions of Z and Z' related by

$$min\{ Z \} = -max\{ Z' \}$$

COMPLICATIONS IN THE SIMPLEX METHODOLOGY

- ☐ Two variables x_j and x_k are tied in the selection of the *nonbasic variable* to replace a current basic variable when $\tilde{c}_j = \tilde{c}_k$; the choice of the new *nonbasic variable* to enter the basis is *arbitrary*
- ☐ Two or more constraints may give rise to the same *minimum ratio value* in selecting the basic variable to be replaced
- ☐ We consider the example of the following tableau

COMPLICATIONS IN THE SIMPLEX METHODOLOGY

$\underline{c}_B$	*basic variables* \ c_j	0	0	0	2	0	$3/2$	*constraint constants*
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_1	1			1	-1	0	2
0	x_2		1		2	0	1	4
0	x_3			1	1	1	1	3
$\tilde{\underline{c}}^T$		0	0	0	$\boxed{2}$	0	$3/2$	$Z = 0$

candidate for basic variable

COMPLICATIONS IN THE SIMPLEX METHODOLOGY

- in selecting the *nonbasic variable* x_4 to enter the basis, we observe that the first two constraints give the same minimum ratio: this means that when x_4 is first increased to 2, both the basic variables x_1 and x_2 will reduce to *0* even though only one of them can become a *nonbasic variable*
- we *arbitrarily* select to remove x_1 from the basis to get the new *basic feasible solution*:

COMPLICATIONS IN THE SIMPLEX METHODOLOGY

$\underline{c}_B$	c_j	*0*	*0*	*0*	2	*0*	3/2	*constraint constants*
	basic variables	x_1	x_2	x_3	x_4	x_5	x_6	
2	x_4	1			1	-1		2
0	x_2	-2	1			2	1	*0*
0	x_3	-1		1		1	1	1
$\tilde{\underline{c}}^T$		-2	*0*	*0*	*0*	*0*	3/2	$Z = 4$

COMPLICATIONS IN THE SIMPLEX METHODOLOGY

- in the new basic feasible solution

$x_1 = 0, \quad x_2 = 0, \quad x_3 = 1, \quad x_4 = 2, \quad x_5 = 0,$ **and** $x_6 = 0$;

we treat x_2 **as a** ***basic variable*** **whose value is** 0;

in effect, x_2 **acts as if it were a** ***nonbasic variable***

DEGENERACY

- ☐ A *degenerate basic feasible* solution is one which has one or more *basic variables* with the value *0*
- ☐ Degeneracy may lead to a number of complications in the simplex approach: an important implication is a minimum ratio of *0* , so that no new *nonbasic variable* may be included in the basis and therefore the basis remains unchanged
- ☐ We consider the following example tableau

COMPLICATIONS IN THE SIMPLEX METHODOLOGY

$\underline{c}_B$	c_j basic variables	0	0	0	2	0	3/2	constraint constants
		x_1	x_2	x_3	x_4	x_5	x_6	
2	x_4		1/2		1		1/2	2
0	x_5	– 1	1/2			1	1/2	0
0	x_3	1	– 1	1			0	1
$\tilde{\underline{c}}^T$		0	– 1	0	0	0	1/2	$Z = 4$

DEGENERACY

the logical choice being the *nonbasic variable* x_6 to enter the basis; this leads to finding the limiting constraint from two equations

$$\frac{1}{2}x_6 = 2 - x_4$$

$$\frac{1}{2}x_6 = 0 - x_5$$

and no constraint in the third equation; thus

$$x_6 = min\{4, 0, \infty\}$$

DEGENERACY

- **Degeneracy may result in the construction of new tableaus without improvement in the objective function value, thereby reducing the efficiency of the computational scheme: in effect, an infinite loop – the so–called *cycling* – is possible**
- **Whenever a tie occurs in the minimum ratio rule, an *arbitrary* decision is made regarding which *basic variable* is replaced, and we ignore the undesirable implications of degeneracy and cycling**

MINIMUM RATIO RULE COMPLICATIONS

- ☐ **The minimum ratio rule may not be able to determine the basic variable to be replaced: this is the case when all equations lead to ∞ as the limit**
- ☐ **Consider the example and corresponding tableau**

$$\max \quad Z = 2x_1 + 3x_2$$

$$s.t.$$

$$\begin{array}{rrrrrrl} x_1 & - & x_2 & + & x_3 & & = 2 \\ -3x_1 & + & x_2 & & & + \; x_4 & = 4 \end{array}$$

$$x_i \geq 0, \qquad i = 1, \dots, 4$$

MINIMUM RATIO RULE COMPLICATIONS

$\underline{c}_B$	c_j	2	3	0	0	*constraint constants*
	basic variables	x_1	x_2	x_3	x_4	
0	x_3	1	-1	1		2
0	x_4	-3	1		1	4
$\tilde{\underline{c}}^T$		2	3	0	0	$Z = 0$

- ☐ **The** *nonbasic variable* x_2 **enters the basis to replace** x_4 **and the new tableau is**

MINIMUM RATIO RULE COMPLICATIONS

$\underline{c}_B$	c_j basic variables	2	3	*0*	*0*	*constraint constants*
		x_1	x_2	x_3	x_4	
0	x_3	– 2		1	1	6
3	x_2	– 3	1		1	4
$\tilde{\underline{c}}^T$		11	*0*	*0*	– 3	$Z = 12$

- ☐ **We select x_1 to enter the basis but we are unable to get limiting constraints from the two equations**

MINIMUM RATIO RULE COMPLICATIONS

$$-2x_1 + x_3 = 6 \qquad x_1 = \frac{1}{2}x_3 - 3$$

$$-3x_1 + x_2 = 4 \qquad x_1 = \frac{1}{3}x_2 - \frac{4}{3}$$

- ☐ In fact, as x_1 increases so do x_2 and x_3 and Z and therefore, the solution is *unbounded*
- ☐ The failure of the minimum ratio rule to result in a bound at any simplex tableau implies that the problem has an *unbounded solution*