ECE 307 – Techniques for Engineering Decisions

3. Introduction to the Simplex Algorithm

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We examine the solution of

\[ A x = b \]

using \textit{Gauss—Jordan} elimination.

We first use a simple example and then generalize to cases of general interest.

Consider the system of two equations in five unknowns:
For this simple example, the number of unknowns exceeds the number of equations and so the system has multiple solutions; this is the principal reason that the LP solution is nontrivial.
The Gauss–Jordan elimination uses elementary row operations:

- multiplication of any equation by a nonzero constant

- addition to any equation of a nonzero constant multiple of any other equation
SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

We transform system $S_1$ by multiplication of equation (i) by $-1$ and its addition to equation (ii) so as to zero out the coefficient of $x_1$ to obtain

$$
S_2 \begin{cases}
 x_1 - 2x_2 + x_3 - 4x_4 + 2x_5 = 2 \\
 x_2 - 2x_3 + x_4 - 3x_5 = 2
\end{cases}
$$
A basic variable is a variable $x_i$ that appears with the coefficient 1 in an equation and with the coefficient 0 in all the other equations.

A variable $x_j$ that is not basic is called a nonbasic variable.

In the system $S$, $x_1$ appears as a basic variable; $x_2, x_3, x_4$ and $x_5$ are nonbasic variables.

Basic variables may be generated through the use of elementary row operations.
DEFINITIONS

- A **pivot operation** is the set of sequential elementary row operations that reduces a system of linear equations into the form in which a specified variable becomes a *basic variable*.

- A **canonical system** is a set of linear equations obtained through *pivot operations* with the property that the system has the same number of *basic variables* as the number of equations in the set.
We transform the system \( S_2 \) into the canonical form of system \( S_3 \):

\[
\begin{align*}
S_3 \left\{
\begin{array}{c}
    x_1 - 3x_3 - 2x_4 - 4x_5 &= 6 \\
    x_2 - 2x_3 + x_4 - 3x_5 &= 2
\end{array}
\right.
\end{align*}
\]

The basic solution is obtained from a canonical system with all the nonbasic variables set to 0.

For the example, we set \( x_3 = x_4 = x_5 = 0 \) and so

\[
x_1 = 6 \quad \text{and} \quad x_2 = 2
\]
A basic feasible solution is a basic solution in which the value of each basic variable is nonnegative.

In the example of system $S_2$, we may choose any two variables to be basic.

In general for a system of $m$ equations in $n$ unknowns there are $\binom{n}{m}$ possible combinations of basic variables.
As $n$ increases, the number of combinations becomes large, even though it remains finite.

For the example, we have

$$\binom{5}{2} = \frac{5!}{3! \cdot 2!} = 10$$

combinations of possible choices.
THE SIMPLEX SOLUTION METHOD

- We next use a simple example to construct the simplex solution method.

- The simplex method is a systematic and efficient scheme to examine a subset of the basic feasible solutions of the LP to hone in on an optimal solution.

- We apply the notions introduced in the definitions we introduced above.
SIMPLEX METHODOLOGY EXAMPLE

\[ \text{max } Z = 5x_1 + 2x_2 + 3x_3 - x_4 + x_5 \]

s.t.

\[ \begin{align*}
\text{canonical form} & \\
\left\{ \begin{array}{l}
x_1 + 2x_2 + 2x_3 + x_4 = 8 \quad (*) \\
3x_1 + 4x_2 + x_3 + x_5 = 7 \quad (**) 
\end{array} \right. \\
x_i \geq 0 \quad i = 1, \ldots, 5
\end{align*} \]
THE SIMPLEX SOLUTION METHOD

- The *canonical form* of the example allows the determination of a basic feasible solution

  \[ x_1 = x_2 = x_3 = 0 \quad x_4 = 8, \quad x_5 = 7 \]

- The corresponding value of the objective is

  \[ Z = -8 + 7 = -1 \]

- The next step is to **improve** the *basic feasible solution* and we need to **find an adjacent basic feasible solution**
An adjacent basic feasible solution is one which differs from the current basic feasible solution in exactly one basic variable.

Note, we characterize a basic feasible solution by the following traits:

\[ \text{basic variable} \geq 0 \]
\[ \text{nonbasic variable} = 0 \]
ADJACENT FEASIBLE SOLUTION

- The search for an adjacent basic feasible solution is based on the idea of the switch of a nonbasic variable into a basic variable by increasing its value from 0 to the largest positive value without the violation of any constraints.

- To make the search efficient, we select the nonbasic variable that improves the value of $Z$ by the largest amount for the maximization objective.
In the example, consider the nonbasic variable $x_1$, we leave $x_2 = x_3 = 0$ and examine the possibility to convert $x_1$ into a basic variable.

The variable $x_1$ enters in both constraints:

\[ x_1 + x_4 = 8 \]

\[ 3x_1 + x_5 = 7 \]
The largest value $x_1$ may assume without making either $x_4$ or $x_5$ negative is

$$\min \left\{ 8, \frac{7}{3} \right\} = \frac{7}{3}$$

We have the new \textit{basic} variable with the value

$$x_1 = \frac{7}{3}$$

and the other \textit{basic} variable is

$$x_4 = \frac{17}{3}$$
and the three *nonbasic* variables are set to 0:

\[ x_2 = x_3 = 0 \text{ and } x_5 = 0 \]

- Note that we obtain an improvement in \( Z \) since its value becomes

\[ Z = 5 \cdot \frac{7}{3} - \frac{17}{3} = \frac{18}{3} = 6 > -1 \]

- We next transform the system of equations into *canonical form*:
SIMPLEX METHODOLOGY EXAMPLE

\[
\begin{align*}
\text{max } Z &= 5x_1 + 2x_2 + 3x_3 - x_4 + x_5 \\
\text{s.t.} \\
\text{non-canonical form for } x_1 & \left\{ 
\begin{array}{l}
x_1 + 2x_2 + 2x_3 + x_4 = 8 \quad (*) \\
3x_1 + 4x_2 + x_3 + x_5 = 7 \quad (**)
\end{array}
\right. \\
x_i & \geq 0 \quad i = 1, \ldots, 5
\end{align*}
\]
ADJACENT FEASIBLE SOLUTION

- Multiply equation (**) by $-\frac{1}{3}$ and add to equation (*):

\[
\frac{2}{3} x_2 + \frac{5}{3} x_3 + x_4 - \frac{1}{3} x_5 = \frac{17}{3}
\]

- Multiply equation (**) by $\frac{1}{3}$:

\[
x_1 + \frac{4}{3} x_2 + \frac{1}{3} x_3 + \frac{1}{3} x_5 = \frac{7}{3}
\]
We continue this process until the condition of optimality is satisfied:

- In a maximization problem, a basic feasible solution is optimal if and only if the relative profits of each nonbasic variable is $\leq 0$.
- In a minimization problem, a basic feasible solution is optimal if and only if the relative costs of each nonbasic variable is $\geq 0$.
The relative profits (costs) are given by the change in $Z$ corresponding to a unit change in a nonbasic variable.

We use this fact to select the next nonbasic variable to enter the basis.
SIMPLEX ALGORITHM FOR MAXIMIZATION

Step 1: start with an initial basic feasible solution with all constraint equations in canonical form.

Step 2: check for optimality condition: if the relative profits are $\leq 0$ for each nonbasic variable, then the basic feasible solution is optimal and stop; else, go to Step 3.
SIMPLEX ALGORITHM FOR MAXIMIZATION

Step 3: select a nonbasic variable to become the new basic variable; check the limits on the nonbasic variable – the limiting constraint determines which basic variable is replaced by the selected nonbasic variable

Step 4: construct the canonical form for the new set of basic variables through elementary row operations; evaluate the basic feasible solution and $Z$ and return to Step 2
THE SIMPLEX TABLEAU

- We use an efficient way to visually represent the steps in the simplex method through a sequence of so-called tableaus.

- We illustrate the tableau for the simple example for the initial basic feasible solution.
### THE SIMPLEX TABLEAU

<table>
<thead>
<tr>
<th>$c_B$</th>
<th>$c_j$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>Constraint Constants</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>7</td>
</tr>
</tbody>
</table>
The optimality check requires the evaluation of

\[ \tilde{c}_j = c_j - \left( \begin{array}{c} \mathbf{c}^T_B \end{array} \right) \cdot \text{column corresponding to } x_j \text{ in canonical form} \]

For each nonbasic variable \( x_j \), for our example, we have

\[
\begin{align*}
\tilde{c}_1 &= 5 - [-1, 1] \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3 \\
\tilde{c}_2 &= 2 - [-1, 1] \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 0 \\
\tilde{c}_3 &= 3 - [-1, 1] \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4
\end{align*}
\]
We interpret each $\tilde{c}_j$ as the change in $Z$ in response to a unit increase in $x_j$.

<table>
<thead>
<tr>
<th>$c_B$</th>
<th>$c_j$</th>
<th>Basic Variables</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>Constraint Constants</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td></td>
<td>$x_4$</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>$x_5$</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>$\tilde{c}^T$</td>
<td></td>
<td>3</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>$Z = -1$</td>
</tr>
</tbody>
</table>

$Z = -1$
SIMPLEX TABLEAU

Note that the optimality test indicates that

\[ \tilde{c}_1 = 3 > 0 \quad \text{and} \quad \tilde{c}_3 = 4 > 0 \]

and so the initial basic feasible solution is not optimal.

Since \( \tilde{c}_3 > \tilde{c}_1 \), we pick \( x_3 \) as the nonbasic variable to enter as a basic variable.

We examine the limiting solution for \( x_3 \) in the two constraint equations:
and so the limiting value is

\[
\min \{4, 7\} = 4
\]

We replace the basic variable \(x_4\) by \(x_3\)
SIMPLEX METHODOLOGY EXAMPLE

\[
\begin{align*}
\text{max } Z &= 5x_1 + 2x_2 + 3x_3 - x_4 + x_5 \\
\text{s.t. } & \\
\text{canonical form in } x_4 \text{ and } x_5
\left\{ \begin{array}{c}
x_1 + 2x_2 + 2x_3 + x_4 = 8 \\
3x_1 + 4x_2 + x_3 + x_5 = 7
\end{array} \right. \\
x_i & \geq 0 \quad i = 1, \ldots, 5
\end{align*}
\]
For the new basic feasible solution, we put the equations into canonical form by

- multiplication of (*) by $\frac{1}{2}$ to obtain (**†)
- subtraction of (**†) from (**) to obtain (***†)

\[
\begin{align*}
\frac{1}{2}x_1 & + x_2 & + x_3 & + \frac{1}{2}x_4 & = 4 & \text{(**†)} \\
\frac{5}{2}x_1 & + 3x_2 & - \frac{1}{2}x_4 & + x_5 & = 3 & \text{(***†)}
\end{align*}
\]

The adjacent basic feasible solution is

\[
x_1 = x_2 = x_4 = 0, \quad x_3 = 4, \quad x_5 = 3
\]
## THE SIMPLEX TABLEAU

<table>
<thead>
<tr>
<th>$c_B$</th>
<th>$c_j$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>(Z = 15)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>5</td>
<td>1/2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1/2</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>5/2</td>
<td>3</td>
<td>–1/2</td>
<td>1</td>
<td></td>
<td>3</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$x_3 = 4$, $x_5 = 3$</td>
</tr>
</tbody>
</table>

\[
\begin{bmatrix}
    c^T \\
    1 & -4 & 0 & -2 & 0
\end{bmatrix}
\]
Since $\tilde{c}_1 > 0$, the basic feasible solution is non-optimal.

We examine how to bring $x_1$ into the basis.

<table>
<thead>
<tr>
<th>equation</th>
<th>limiting basic variable</th>
<th>upper limit on $x_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>($*\dagger$)</td>
<td>$x_3$</td>
<td>$4/(1/2) = 8$</td>
</tr>
<tr>
<td>($* * \dagger$)</td>
<td>$x_5$</td>
<td>$3/(5/2) = 6/5$</td>
</tr>
</tbody>
</table>
The variable \( x_1 \) enters the basis with the value

\[
\min \left\{ 8, \frac{6}{5} \right\} = \frac{6}{5}
\]

and \( x_5 \) is replaced as a basic variable by \( x_1 \)

We need to put the equations

\[
\begin{align*}
\frac{1}{2}x_1 + x_2 + x_3 + \frac{1}{2}x_4 &= 4 \quad (\star \dagger) \\
\frac{5}{2}x_1 + 3x_2 - \frac{1}{2}x_4 + x_5 &= 3 \quad (** \dagger)
\end{align*}
\]

into canonical form for the basic variables \( x_3 \) and \( x_1 \)
The following elementary row operations are used

- multiply \((**\uparrow)\) by \(-1/5\) and add to \((*\uparrow)\)

\[
\frac{2}{5}x_2 + x_3 + \frac{3}{5}x_4 - \frac{1}{5}x_5 = \frac{17}{5}
\]

- multiply \((**\uparrow)\) by \(2/5\)

\[
x_1 + \frac{6}{5}x_2 - \frac{1}{5}x_4 + \frac{2}{5}x_5 = \frac{6}{5}
\]

and construct the corresponding tableau
## THE SIMPLEX TABLEAU

<table>
<thead>
<tr>
<th>$c_B$</th>
<th>$c_j$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>constraint constants</th>
</tr>
</thead>
<tbody>
<tr>
<td>basic variables</td>
<td>5</td>
<td>2</td>
<td>3</td>
<td>$-1$</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$x_3$</td>
<td>2/5</td>
<td>1</td>
<td>3/5</td>
<td>$-1/5$</td>
<td>17/5</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$x_1$</td>
<td>1</td>
<td>6/5</td>
<td>$-1/5$</td>
<td>2/5</td>
<td>6/5</td>
<td></td>
</tr>
<tr>
<td>$\tilde{c}^T$</td>
<td>0</td>
<td>$-26/5$</td>
<td>0</td>
<td>$-9/5$</td>
<td>$-2/5$</td>
<td>$Z = 81/5$</td>
<td></td>
</tr>
</tbody>
</table>

$\tilde{c}_j \leq 0$ implies optimality

16.2 > 15
SIMPLEX TABLEAU EXAMPLE

\[ \text{max } Z = 3x_1 + 2x_2 \]

\[ \text{s.t. } \]
\[ -x_1 + 2x_2 \leq 4 \]
\[ 3x_1 + 2x_2 \leq 14 \]
\[ x_1 - x_2 \leq 3 \]
\[ x_1 \geq 0 \quad x_2 \geq 0 \]
We put this problem into standard form:

\[
\begin{align*}
\text{max } Z &= 3x_1 + 2x_2 \\
\text{s.t.} & \quad -x_1 + 2x_2 + x_3 = 4 \\
& \quad 3x_1 + 2x_2 + x_4 = 14 \\
& \quad x_1 - x_2 + x_5 = 3 \\
& \quad x_1, \ldots, x_5 \geq 0
\end{align*}
\]

\(x_3, x_4, x_5\) are fictitious – or slack – variables
SIMPLEX TABLEAU EXAMPLE

<table>
<thead>
<tr>
<th>$c_B$</th>
<th>basic variables</th>
<th>$c_j$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>constraint constants</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$x_3$</td>
<td>-1</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>$x_4$</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>14</td>
</tr>
<tr>
<td>0</td>
<td>$x_5$</td>
<td>1</td>
<td>-1</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>$\tilde{c}^T$</td>
<td></td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td>$Z = 0$</td>
</tr>
</tbody>
</table>

$$\tilde{c}_j = c_j - (c_B^T \cdot \text{column corresponding to } x_j)$$
SIMPLEX TABLEAU EXAMPLE

- The data in $\tilde{c}^T$ indicate that the highest relative profits correspond to $x_1$ and so we wish to make $x_1$ a basic variable.

- To bring $x_1$ into the basis requires to evaluate

  $$\min \left\{ \infty, \frac{14}{3}, 3 \right\} = 3$$

  and so $x_1$ replaces $x_5$ with the value 3.

- We evaluate the basic variable at the adjacent basic feasible solution and convert into canonical form; the new tableau becomes...
# SIMPLEX TABLEAU EXAMPLE

<table>
<thead>
<tr>
<th>$c_B$</th>
<th>$c_j$</th>
<th>Basic Variables</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>Constraint Constants</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>$x_3$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>0</td>
<td></td>
<td>$x_4$</td>
<td>5</td>
<td></td>
<td>1</td>
<td></td>
<td>-3</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>$x_1$</td>
<td>1</td>
<td>-1</td>
<td></td>
<td>1</td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>$\tilde{c}^T$</td>
<td></td>
<td>$c_j$</td>
<td>0</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>-3</td>
<td>$Z = 9$</td>
</tr>
</tbody>
</table>
We reproduce here the calculation of the \( \tilde{c}^T \) components

\[
\tilde{c}_j = c_j - \left( c^T_B \cdot \text{column corresponding to } x_j \right)
\]

for each nonbasic variable \( x_j \)

Note that, by definition, \( \tilde{c}_i = 0 \) for each basic variable \( x_i \)
SIMPLEX TABLEAU EXAMPLE

The calculations give

\[ \tilde{c}_1 = 0 \text{ by definition since } x_1 \text{ is in the basis} \]

\[ \tilde{c}_2 = 2 - \left[ \begin{array}{ccc} 0 & 0 & 3 \\ 1 & 5 & -1 \end{array} \right] = 5 \]

\[ \text{indicates possible improvement} \]

\[ \tilde{c}_3 = 0 \text{ by definition since } x_3 \text{ is in the basis} \]

\[ \tilde{c}_4 = 0 \text{ by definition since } x_4 \text{ is in the basis} \]

\[ \tilde{c}_5 = 0 - \left[ \begin{array}{ccc} 0 & 0 & 3 \\ 1 & -3 & 1 \end{array} \right] = -3 \]
Clearly, the only choice is to get $x_2$ into the basis and so we need to establish the limiting condition from the three equations by evaluating

$$\min \{7, 1, \infty\} = 1$$

and so $x_2$ replaces $x_4$, which becomes a nonbasic variable.

We need to rewrite the equations into canonical form for $x_3$ and $x_2$ and construct the new tableau.
### SIMPLEX TABLEAU EXAMPLE

<table>
<thead>
<tr>
<th>$c_B$</th>
<th>$c_j$</th>
<th>$c_T$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>constraint constants</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$x_3$</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td>$-1/5$</td>
<td>$8/5$</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>$x_2$</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td>$1/5$</td>
<td>$-3/5$</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>$x_1$</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td>$1/5$</td>
<td>$2/5$</td>
<td>4</td>
</tr>
</tbody>
</table>

**Basic variables**

- $x_1$ (1st row)
- $x_2$ (2nd row)
- $x_3$ (3rd row)

**Objective Function**

$$ Z = 14 $$

**Optimal Condition**

$$ \tilde{c}_j \leq 0 \ \forall \ j \Rightarrow \text{optimum} $$
SIMPLEX TABLEAU EXAMPLE

- An optimum is at the solution of

\[
\begin{align*}
    x_3 & - \frac{1}{5} x_4 + \frac{8}{5} x_5 = 6 \\
    x_2 & + \frac{1}{5} x_4 - \frac{2}{5} x_5 = 1 \\
    x_1 & + \frac{1}{5} x_4 + \frac{2}{5} x_5 = 4
\end{align*}
\]
This optimum is given by

\[ x_4 = x_5 = 0 \]

\[ x_3 = 6 \]

\[ x_2 = 1 \]

\[ x_1 = 4 \]
Consider the following LP

\[
\begin{align*}
\text{max } Z & \quad = \quad 3x_1 + 2x_2 \\
\text{s.t.} & \quad -x_1 + 2x_2 \leq 4 \\
& \quad 3x_1 + 2x_2 \leq 14 \\
& \quad x_1 - x_2 \leq 3 \\
& \quad x_1 \geq 0, \quad x_2 \geq 0
\end{align*}
\]
LINEAR PROGRAMMING EXAMPLE

A

E

D

B

C

x_1

3x_1 + 2x_2 = 4

-x_1 + 2x_2 = 3

3x_1 + 2x_2 = 14

3x_1 + 2x_2 = 9

x_2

tableau 1

tableau 2

tableau 3
The tableau approach leads to $C$ which is an optimal solution with

$$x_1 = 4, \ x_2 = 1, \ x_3 = 6, \ x_4 = 0, \ x_5 = 0$$

Note that any point along $CD$ has $Z = 14$ and as such $D$ is another optimal solution corresponding to an adjacent basic feasible solution.

We may obtain $D$ from $C$ by bringing into the basis the nonbasic variable $x_5$ in Tableau 3; note that $\tilde{c}_5 = 0$. 
We may choose $x_5$ as a basic variable without affecting $Z$ since its relative profits are 0; we compute the limiting value of $x_5$

The limit is imposed by $x_3$ which, consequently, leaves the basis

The corresponding tableau is:
LINEAR PROGRAMMING EXAMPLE

<table>
<thead>
<tr>
<th>$c_B$</th>
<th>$c_j$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>constraint constants</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$x_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>15/4</td>
</tr>
<tr>
<td>2</td>
<td>$x_2$</td>
<td>1</td>
<td>3/8</td>
<td>1/8</td>
<td>0</td>
<td>0</td>
<td>13/4</td>
</tr>
<tr>
<td>3</td>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
<td>$-1/4$</td>
<td>1/4</td>
<td>0</td>
<td>5/2</td>
</tr>
<tr>
<td>$\tilde{c}^T$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-1$</td>
<td>0</td>
<td>$Z = 14$</td>
<td></td>
</tr>
</tbody>
</table>

$\tilde{c}_j \leq 0 \quad \forall \ j$
The adjacent feasible solution is given by

\[ x_1 = \frac{5}{2}, \quad x_2 = \frac{13}{4}, \quad x_3 = x_4 = 0, \quad x_5 = \frac{15}{4} \]

Note that at this basic feasible solution,

\[ \bar{c}_j \leq 0 \quad \forall \, j \]

and so this is also an optimal solution.
In general, an *alternate optimal solution* is indicated whenever there exists a *nonbasic variable* $x_j$ with $\tilde{c}_j = 0$

in an optimal tableau; such a situation indicates a *non unique optimum* for the *LP*
Consider a minimization LP with the form given by

\[
\begin{align*}
\text{min} \quad & Z = \sum_{i=1}^{n} c_i x_i \\
\text{s.t.} \quad & Ax = b \\
& x \geq 0
\end{align*}
\]
MINIMIZATION \( LP \)

- We replace the optimality check in the simplex scheme by the minimization optimality check:

  If each coefficient \( \tilde{c}_j \geq 0 \), stop; else, select the nonbasic variable with the most negative–valued \( \tilde{c} \) component to become the new basic variable.
Every minimization LP may be solved as a maximization LP because of equivalence

\[ \min Z = c^T x \quad \text{max} \quad Z' = (-c^T) x \]
\[ \text{s.t.} \quad Ax = b \quad \text{s.t.} \quad Ax = b \]
\[ x \geq 0 \quad x \geq 0 \]

with the solutions of \( Z \) and \( Z' \) related by

\[ \min \{ Z \} = -\max \{ Z' \} \]
COMPLICATIONS IN THE SIMPLEX METHODOLOGY

- Two variables $x_j$ and $x_k$ are tied in the selection of the nonbasic variable to replace a current basic variable when $\tilde{c}_j = \tilde{c}_k$; the choice of the new nonbasic variable to enter the basis is arbitrary.

- Two or more constraints may give rise to the same minimum ratio value in selecting the basic variable to be replaced.

- We consider the example of the following tableau.
### COMPLICATIONS IN THE SIMPLEX METHODOLOGY

<table>
<thead>
<tr>
<th>( c_B )</th>
<th>( c_j )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
<th>( x_6 )</th>
<th>( constraint )</th>
<th>( constants )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( x_1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>2</td>
<td>( constraint )</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>( x_2 )</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>( constraint )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>( x_3 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>( constraint )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ \tilde{c}^T = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 3/2 \end{bmatrix} \]

\[ Z = 0 \]

**candidate for basic variable**
in selecting the nonbasic variable $x_4$ to enter the basis, we observe that the first two constraints give the same minimum ratio: this means that when $x_4$ is first increased to 2, both the basic variables $x_1$ and $x_2$ will reduce to 0 even though only one of them can become a nonbasic variable.

we *arbitrarily* select to remove $x_1$ from the basis to get the new basic feasible solution:
## COMPLICATIONS IN THE SIMPLEX METHODOLOGY

<table>
<thead>
<tr>
<th>$c_B$</th>
<th>$c_j$</th>
<th>$c^T$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>Constraint constants</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$x_4$</td>
<td>1</td>
<td></td>
<td>1</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>$x_2$</td>
<td>$-2$</td>
<td>1</td>
<td></td>
<td>2</td>
<td></td>
<td>1</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>$x_3$</td>
<td>$-1$</td>
<td></td>
<td>1</td>
<td></td>
<td>1</td>
<td></td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

$\tilde{c}^T = \begin{bmatrix} -2 & 0 & 0 & 0 & 3/2 \end{bmatrix}$

$Z = 4$
COMPLICATIONS IN THE SIMPLEX METHODOLOGY

○ in the new basic feasible solution

\[ x_1 = 0, \quad x_2 = 0, \quad x_3 = 1, \quad x_4 = 2, \quad x_5 = 0, \text{ and } x_6 = 0; \]

we treat \( x_2 \) as a basic variable whose value is 0;

in effect, \( x_2 \) acts as if it were a nonbasic variable
A degenerate basic feasible solution is one which has one or more basic variables with the value 0.

Degeneracy may lead to a number of complications in the simplex approach: an important implication is a minimum ratio of 0, so that no new nonbasic variable may be included in the basis and therefore the basis remains unchanged.

We consider the following example tableau...
### COMPLICATIONS IN THE SIMPLEX METHODOLOGY

<table>
<thead>
<tr>
<th>$c_B$</th>
<th>$c_j$</th>
<th>$0$</th>
<th>$0$</th>
<th>$0$</th>
<th>$2$</th>
<th>$0$</th>
<th>$3/2$</th>
<th>constraint constants</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$x_3$</td>
<td>$x_4$</td>
<td>$x_5$</td>
<td>$x_6$</td>
<td></td>
</tr>
<tr>
<td>$2$</td>
<td>$x_4$</td>
<td>$1/2$</td>
<td>$1$</td>
<td>$1/2$</td>
<td>$1/2$</td>
<td>$1/2$</td>
<td>$2$</td>
<td></td>
</tr>
<tr>
<td>$0$</td>
<td>$x_5$</td>
<td>$-1$</td>
<td>$1/2$</td>
<td>$1$</td>
<td>$1/2$</td>
<td>$1/2$</td>
<td>$0$</td>
<td></td>
</tr>
<tr>
<td>$0$</td>
<td>$x_3$</td>
<td>$1$</td>
<td>$-1$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$1$</td>
<td></td>
</tr>
<tr>
<td>$\tilde{c}^T$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$1/2$</td>
<td>$Z = 4$</td>
<td></td>
</tr>
</tbody>
</table>

The table shows the basic variables and the constraint constants for a specific simplex method problem. The basic variables are $x_4$, $x_5$, and $x_3$, and the constraint constants are calculated accordingly.
the logical choice being the *nonbasic variable* $x_6$ to enter the basis; this leads to finding the limiting constraint from two equations

\[
\frac{1}{2} x_6 = 2 - x_4
\]

\[
\frac{1}{2} x_6 = 0 - x_5
\]

and no constraint in the third equation; thus

\[
x_6 = min\{4, 0, \infty\}
\]
Degeneracy may result in the construction of new tableaus without improvement in the objective function value, thereby reducing the efficiency of the computational scheme: in effect, an infinite loop – the so-called cycling – is possible.

Whenever a tie occurs in the minimum ratio rule, an arbitrary decision is made regarding which basic variable is replaced, and we ignore the undesirable implications of degeneracy and cycling.
The minimum ratio rule may not be able to determine the basic variable to be replaced: this is the case when all equations lead to $\infty$ as the limit.

Consider the example and corresponding tableau:

\[
\begin{align*}
\text{max } & \quad Z = 2x_1 + 3x_2 \\
\text{s.t. } & \quad x_1 - x_2 + x_3 = 2 \\
& \quad -3x_1 + x_2 + x_4 = 4 \\
& \quad x_i \geq 0, \quad i = 1, \ldots, 4
\end{align*}
\]
MINIMUM RATIO RULE

COMPLICATIONS

<table>
<thead>
<tr>
<th>$c_B$</th>
<th>basic variables</th>
<th>$c_j$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>constraint constants</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$x_3$</td>
<td></td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>$x_4$</td>
<td></td>
<td>-3</td>
<td>1</td>
<td>1</td>
<td></td>
<td>4</td>
</tr>
</tbody>
</table>

$\tilde{c}^T$

$\begin{array}{cccccc}
2 & 3 & 0 & 0 & \hline
\end{array}$

$Z = 0$

- The nonbasic variable $x_2$ enters the basis to replace $x_4$ and the new tableau is

- $x_4$ and the new tableau is
MINIMUM RATIO RULE

COMPLICATIONS

<table>
<thead>
<tr>
<th>( c_B )</th>
<th>( c_j )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>constraint ( \text{constants} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( x_3 )</td>
<td>–2</td>
<td>1</td>
<td>1</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( x_2 )</td>
<td>–3</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>( \tilde{c}^T )</td>
<td>11</td>
<td>0</td>
<td>0</td>
<td>–3</td>
<td>( Z = 12 )</td>
<td></td>
</tr>
</tbody>
</table>

We select \( x_1 \) to enter the basis but we are unable to get limiting constraints from the two equations.
In fact, as $x_1$ increases so do $x_2$ and $x_3$ and $Z$ and therefore, the solution is unbounded.

The failure of the minimum ratio rule to result in a bound at any simplex tableau implies that the problem has an unbounded solution.