Lecture 2. Introduction to Linear Programming

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OUTLINE

- The nature of a programming or an optimization problem
- Linear programming (LP): salient characteristics
- The LP problem formulation
- The LP problem solution
- Extensive illustrations with numerical examples
EXAMPLE 1: HIGH/LOW HEEL SHOE CHOICE PROBLEM

- A lady is headed to a party and is trying to find a pair of shoes to wear; the choice is narrowed down to two possible choices:
  - a high heel pair; and
  - a low heel pair

- The high heel shoes look more beautiful but are not as comfortable as the competing pair

- Which pair should she choose?
We first quantify our assessment along the two dimensions of *looks* and *comfort* in a table.

<table>
<thead>
<tr>
<th>aspect</th>
<th>maximum value</th>
<th>assessment</th>
<th>weighting factor (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>high heels</td>
<td>low heels</td>
</tr>
<tr>
<td>aesthetics</td>
<td>5.0</td>
<td>4.2</td>
<td>3.6</td>
</tr>
<tr>
<td>comfort</td>
<td>5.0</td>
<td>3.5</td>
<td>4.8</td>
</tr>
</tbody>
</table>

Next, we represent the decision in terms of two decision variables:
MODEL FORMULATION

\[ x_H = \begin{cases} 
1 & \text{choose high} \\
0 & \text{otherwise} 
\end{cases} \quad x_L = \begin{cases} 
1 & \text{choose low} \\
0 & \text{otherwise} 
\end{cases} \]

- We formulate the objective to be the maximization of the weighted assessment

\[ \max \{ 70\% \ast \text{aesthetics} + 30\% \ast \text{comfort} \} \]

- We state the objective in terms of the defined decision variables

\[ \max Z = x_H [(4.2)(0.7) + (3.5)(0.3)] + x_L [(3.6)(0.7) + (4.8)(0.3)] \]
Next, we consider the problem constraints:

- only one pair of shoes can be selected
- each decision variable is nonnegative

We express the constraints in terms of $x_H$ and $x_L$:

\[ x_H + x_L = 1 \]
\[ x_H \geq 0, \quad x_L \geq 0 \]
Decision variables:

\[ x_H = \begin{cases} 
1 & \text{choose high} \\
0 & \text{otherwise} 
\end{cases} \quad \quad \quad x_L = \begin{cases} 
1 & \text{choose low} \\
0 & \text{otherwise} 
\end{cases} \]

Objective function:

\[ \max Z = 3.99x_H + 3.96x_L \]

Constraints:

\[ x_H + x_L = 1 \]

\[ x_H \geq 0, \quad x_L \geq 0 \]
We determine the values $x^*_H$ and $x^*_L$ which result in the value of $Z^*$ such that

$$Z^* = Z\left(x^*_H, x^*_L\right) \geq Z\left(x^*_H, x^*_L\right) \quad (\dagger)$$

for all feasible $\left(x^*_H, x^*_L\right)$.

We call such a solution an optimal solution.

A feasible solution is one that satisfies all the constraints on the problem.

The optimal solution, denoted by $\left(x^*_H, x^*_L\right)$, is selected from all the feasible solutions to the problem so as to satisfy $(\dagger)$.
We enumerate all the feasible solutions: in this problem there are only two alternatives:

\[ A: \begin{cases} x_H = 1 \\ x_L = 0 \end{cases} \quad B: \begin{cases} x_H = 0 \\ x_L = 1 \end{cases} \]

We evaluate \( Z \) for \( A \) and \( B \) and compare

\[ Z_A = 3.99 \quad \text{and} \quad Z_B = 3.96 \]

so that \( Z_A > Z_B \) and so \( A \) is the optimal choice.

The optimal solution is

\[ x^*_H = 1, \quad x^*_L = 0 \quad \text{and} \quad Z^* = 3.99 \]
CHARACTERISTICS OF A PROGRAMMING/OPTIMIZATION PROBLEM

- The objective is to select the decision among the various alternatives and therefore requires first the definition of the decision variables.

- We determine the “best” decision simply based on the objective function value; to do so we require the mathematical formulation of the objective function.

- The decision must satisfy each specified constraint and so we require the mathematical statement of the problem constraints.
The problem statement is characterized by:

- Decision variables: continuous valued, integer valued
- Objective function: linear, non-linear
- Constraints: linear, non-linear
PROGRAMMING PROBLEM CLASSES

- Linear/nonlinear programming
- Static/dynamic programming
- Integer programming
- Mixed programming
A company is producing two types of conductors for **EHV** transmission lines. The supply department can provide up to 1 ton of metal each day. We schedule the production so as to maximize the profits of the company.

<table>
<thead>
<tr>
<th>type</th>
<th>conductor</th>
<th>production capacity (unit/day)</th>
<th>metal needed (tons/unit)</th>
<th>profits ($/unit)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>ACSR 84/19</td>
<td>4</td>
<td>1/6</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>ACSR 18/7</td>
<td>6</td>
<td>1/9</td>
<td>5</td>
</tr>
</tbody>
</table>
Formulation of the objective: to maximize the profits of the company

Means to attain this objective: determine how many units of product 1 and of product 2 to produce each day

Consideration of all the constraints: the daily production capacity limits, the daily metal supply limit and common sense requirements
MODEL CONSTRUCTION

- We define the decision variables to be
  \[ x_1 = \text{number of type 1 units produced per day} \]
  \[ x_2 = \text{number of type 2 units produced per day} \]

- We define the objective to be
  \[ Z = \text{profits ($/day)} \]
  \[ = 3x_1 + 5x_2 \]

- Sanity check for units of the objective function
  \[ ($/day) = ($/unit) \cdot (unit/day) \]
Objective function:

\[ \text{max } Z = 3x_1 + 5x_2 \]

Constraints:

- capacity limits:
  \[ x_1 \leq 4 \quad x_2 \leq 6 \]

- metal supply limit:
  \[ \frac{x_1}{6} + \frac{x_2}{9} \leq 1 \]

- common sense requirements:
  \[ x_1 \geq 0 \quad x_2 \geq 0 \]
PROBLEM STATEMENT

\[
\text{max } Z = 3x_1 + 5x_2
\]

s.t.

\[
x_1 \leq 4
\]

\[
x_2 \leq 6
\]

\[
\frac{x_1}{6} + \frac{x_2}{9} \leq 1
\]

\[
x_1 \geq 0 \ , \ x_2 \geq 0
\]
VISUALIZATION OF THE FEASIBLE REGION

\[ x_1 \geq 0 \quad , \quad x_1 \leq 4 \quad , \quad x_2 \geq 0 \]
VISUALIZATION OF THE FEASIBLE REGION

\[ x_1 \geq 0, \ x_2 \leq 6, \ x_2 \geq 0 \]
VISUALIZATION OF THE FEASIBLE REGION

\[ \frac{x_1}{6} + \frac{x_2}{9} \leq 1 \]

\[ x_1 \geq 0, \ x_2 \geq 0 \]
THE FEASIBLE REGION

The feasible region is defined by the set of points $(x_1, x_2)$ that satisfy the constraints:

1. $x_1 = 4$
2. $x_2 = 6$
3. $(0, 6)$
4. $(2, 6)$
5. $(4, 3)$
6. $(4, 0)$
7. $(0, 0)$

The feasible region is the blue shaded area on the graph.
FEASIBLE SOLUTION SPACE

\[ Z = 3x_1 + 5x_2 \]

Constraints:

- \[ x_1 \geq 4 \]
- \[ x_2 \geq 2 \]
- \[ x_1 \leq 6 \]
- \[ x_2 \leq 6 \]

Optimal Solution:

- \[ Z = 27 \text{ at } (1.5, 4.5) \]
- \[ Z = 16 \text{ at } (2, 2) \]

Graphically represented with points:

- \( (0, 0) \)
- \( (0, 6) \)
- \( (4, 0) \)
- \( (4, 3) \)
- \( (2, 6) \)

The shaded area represents the feasible solution space.
CONTOURS OF CONSTANT \( z \)

\[
\begin{align*}
\max Z &= 3x_1 + 5x_2 \\
Z &= 36 \\
\frac{x_1}{6} + \frac{x_2}{9} &= 1
\end{align*}
\]
For this simple problem, we can graphically obtain the optimal solution.

The optimal solution of this problem is:

\[ x_1^* = 2 \quad \text{and} \quad x_2^* = 6 \]

The objective value at the optimal solution is

\[ Z^* = 3x_1^* + 5x_2^* = 36 \]
A linear programming problem is an *optimization problem* with a *linear* objective function and *linear* constraints.

*problem with a linear objective function and linear constraints.*
EXAMPLE 3: ONE-POTATO, TWO-POTATO PROBLEM

- Mr. Spud manages the Potatoes-R-Us Co. which processes potatoes into packages of freedom fries \((F)\), hash browns \((H)\) and chips \((C)\).
- Mr. Spud can buy potatoes from two sources; each source has distinct characteristics/limits.
- The problem is to determine the respective quantities Mr. Spud needs to buy from source 1 and from source 2 so as to maximize his profits.
EXAMPLE 3: ONE-POTATO, TWO-POTATO PROBLEM

The given data are summarized in the table:

<table>
<thead>
<tr>
<th>product</th>
<th>source 1 uses (%)</th>
<th>source 2 uses (%)</th>
<th>sales limit (tons)</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>20</td>
<td>30</td>
<td>1.8</td>
</tr>
<tr>
<td>H</td>
<td>20</td>
<td>10</td>
<td>1.2</td>
</tr>
<tr>
<td>C</td>
<td>30</td>
<td>30</td>
<td>2.4</td>
</tr>
<tr>
<td>profits ($/ton)</td>
<td>5</td>
<td>6</td>
<td>–</td>
</tr>
</tbody>
</table>

The following assumptions hold:

- 30% waste for each source
- Production may not exceed the sales limit
Decision variables:

\[ x_1 = \text{quantity purchased from source 1} \]
\[ x_2 = \text{quantity purchased from source 2} \]

Objective function:

\[ \max Z = 5x_1 + 6x_2 \]

Constraints:

\[ 0.2x_1 + 0.3x_2 \leq 1.8 \quad (F) \]
\[ 0.2x_1 + 0.1x_2 \leq 1.2 \quad (H) \quad x_1 \geq 0, x_2 \geq 0 \]
\[ 0.3x_1 + 0.3x_2 \leq 2.4 \quad (C) \]
FEASIBLE REGION DETERMINATION

freedom fries $F$

hash browns $H$

chips $C$
THE FEASIBLE REGION

feasible region
EXAMPLE 3: CONTOURS OF CONSTANT $Z$

$max Z = 5x_1 + 6x_2$

Contour lines for $Z = 18, 24, 30, 36, 40.5$.

Optimal solution at $(4.5, 3)$. 

The maximum value of $Z$ is achieved at this point.
The optimal solution of this problem is:

\[ x_1^* = 4.5 \quad x_2^* = 3 \]

The objective value at the optimal solution is:

\[ Z^* = 5x_1^* + 6x_2^* = 40.5 \]
**IMPORTANT OBSERVATIONS**

- Constant $Z$ lines are parallel and change monotonically along the direction normal to the contours of constant values of $Z$.

- An *optimal* solution must be at one of the *corner points* of the feasible region: fortuitously, there are only a *finite* number of corner points.

- If a particular corner point gives a better solution (in terms of its $Z$ value) than that at every other adjacent corner point, then, it is an *optimal* solution.
CONCEPTUAL SOLUTION PROCEDURE

- Initialization step: start at a corner point
- Iteration step: move to an improved adjacent corner point and repeat this step as many times as needed
- Stopping rule: stop when the corner point solution is better than that at each adjacent corner point
- This conceptual procedure forms the basis of the simplex approach
EXAMPLE 3: THE SIMPLEX APPROACH SOLUTION

\[ \text{max } Z = 5x_1 + 6x_2 \]

\[ Z = 36 \]
\[ (0, 6) \]

\[ Z = 40.5 \]
\[ (4.5, 3) \]

\[ Z = 30 \]
\[ (6, 0) \]
EXAMPLE 3: THE SIMPLEX APPROACH SOLUTION

<table>
<thead>
<tr>
<th>step</th>
<th>$x_2$</th>
<th>$x_1$</th>
<th>$Z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>6</td>
<td>36</td>
</tr>
<tr>
<td>2</td>
<td>4.5</td>
<td>3</td>
<td>40.5</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>0</td>
<td>30</td>
</tr>
</tbody>
</table>
EXAMPLE 3: THE SIMPLEX APPROACH SOLUTION

\[ Z = 40.5 \quad \text{at (4.5,3)} \]

\[ Z = 30 \quad \text{at (6,0)} \]

\[ Z = 36 \quad \text{at (0,6)} \]

\[ Z = 0 \quad \text{at (0,0)} \]

\[ \max Z = 5x_1 + 6x_2 \]
EXAMPLE 3: THE SIMPLEX APPROACH SOLUTION

1. Start at \((0,0)\) with \(Z(0,0) = 0\)

2. (i) Move from \((0,0)\) to \((0,6)\), \(Z(0,6) = 36\)

   (ii) Move from \((0,6)\) to \((4.5,3)\); compute \(Z(4.5,3) = 40.5\)

3. Compare the objective at \((4.5,3)\) to values at \((6,0)\)
   and at \((0,6)\):

   \[
   Z(4.5,3) \geq Z(6,0) \\
   Z(4.5,3) \geq Z(0,6)
   \]

   therefore, \((4.5,3)\) is optimal
Key requirements of a programming problem:

1. To make a decision, we must define the **decision variables**.
2. To achieve the specified objective, we must express mathematically the **objective function**.
3. To ensure **feasibility**, the decision variables must satisfy each **mathematically formulated constraint**.
REVIEW

☐ Key attributes of an LP

☐ the objective function is linear

☐ the constraints are linear

☐ Basic steps in formulating a programming problem

☐ definition of decision variables

☐ statement of the objective function

☐ formulation of the constraints
REVIEW

- Words of caution: care is required with units and attention is needed to not ignore the implicit constraints, such as nonnegativity, and the common sense requirements in an LP formulation.

- Graphical solution approach for two-variable problems:
  - feasible region determination
  - contours of constant $Z$
  - identification of the vertex with optimal $Z^*$
There are 8 grade 1 and 10 grade 2 inspectors available for QC inspection; at least 1,800 pieces must be inspected in each 8-hour day.

Problem data are summarized below:

<table>
<thead>
<tr>
<th>grade level</th>
<th>speed (unit/h)</th>
<th>accuracy (%)</th>
<th>wages ($/h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25</td>
<td>98</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td>95</td>
<td>3</td>
</tr>
</tbody>
</table>
EXAMPLE 4: INSPECTION OF GOODS PRODUCED

- Each error costs $2

- The problem is to determine the optimal assignment of inspectors, i.e., the number of inspectors of grade 1 and that of grade 2 to result in the least-cost QC inspection effort
EXAMPLE 4: FORMULATION

Definition of decision variables:

\[ x_1 = \text{number of grade 1 inspectors assigned} \]

\[ x_2 = \text{number of grade 2 inspectors assigned} \]

Objective function

- optimal assignment: minimum costs

- costs = wages + errors
EXAMPLE 4: FORMULATION

• each grade 1 inspector costs:

\[ 4 + 2 (25)(0.02) = 5 \, \text{\$} / h \]

• each grade 2 inspector costs:

\[ 3 + 2 (15)(0.05) = 4.5 \, \text{\$} / h \]

• total daily inspection costs in \$ are

\[ Z = 8[5x_1 + 4.5x_2] = 40x_1 + 36x_2 \] \text{\$(} \]
EXAMPLE 4: FORMULATION

Constraints:

- job completion:
  \[ 8(25)x_1 + 8(15)x_2 \geq 1,800 \]
  \[ \Leftrightarrow 200x_1 + 120x_2 \geq 1,800 \]
  \[ \Leftrightarrow 5x_1 + 3x_2 \geq 45 \]

- availability limit:
  \[ x_1 \leq 8 \]
  \[ x_2 \leq 10 \]

- nonnegativity:
  \[ x_1 \geq 0, \ x_2 \geq 0 \]
EXAMPLE 4: PROBLEM STATEMENT

SUMMARY

- **Decision variables:**
  
  \[ x_1 = \text{number of grade 1 inspectors assigned} \]
  \[ x_2 = \text{number of grade 2 inspectors assigned} \]

- **Objective function:**
  \[ \text{min } Z = 40x_1 + 36x_2 \]

- **Constraints:**
  \[ 5x_1 + 3x_2 \geq 45 \]
  \[ x_1 \leq 8 \]
  \[ x_2 \leq 10 \]
  \[ x_1 \geq 0, \ x_2 \geq 0 \]
MULTI – PERIOD SCHEDULING

- More than one period is involved
- The result of each period affects the initial conditions for the next period and therefore the solution
- We need to define variables to take into account the initial conditions in addition to the decision variables of the problem
EXAMPLE 5: HYDROELECTRIC POWER SYSTEM OPERATIONS

- We consider a single operator of a system consisting of two water reservoirs with a hydroelectric plant attached to each reservoir.
- We schedule the two power plant operations over a two-period horizon.
- We are interested in the plan that maximizes the total revenues of the system operator.
EXAMPLE 5: HYDROELECTRIC POWER SYSTEM OPERATIONS

flows of water in the system

res A inflow

plant A

res B inflow

plant B

flows of water in the system
### EXAMPLE 5: kAft RESERVOIR DATA

<table>
<thead>
<tr>
<th>parameter</th>
<th>reservoir A</th>
<th>reservoir B</th>
</tr>
</thead>
<tbody>
<tr>
<td>maximum capacity</td>
<td>2,000</td>
<td>1,500</td>
</tr>
<tr>
<td>predicted inflow in period 1</td>
<td>200</td>
<td>40</td>
</tr>
<tr>
<td>predicted inflow in period 2</td>
<td>130</td>
<td>15</td>
</tr>
<tr>
<td>minimum allowable level</td>
<td>1,200</td>
<td>800</td>
</tr>
<tr>
<td>level at start of period 1</td>
<td>1,900</td>
<td>850</td>
</tr>
</tbody>
</table>
EXAMPLE 5: SYSTEM CHARACTERISTICS

plant A → 1 kA\text{f} → \text{plant A} → 400 \text{ MWh}

plant B → 1 kA\text{f} → \text{plant B} → 200 \text{ MWh}

<table>
<thead>
<tr>
<th>reservoir</th>
<th>max kA\text{f} for generation per period</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>150</td>
</tr>
<tr>
<td>B</td>
<td>87.5</td>
</tr>
</tbody>
</table>
EXAMPLE 5: SYSTEM CHARACTERISTICS

- Two-tier price for the $MWh$ demand in each period
  - up to 50,000 $MWh$ can be sold @ 20 $/MWh$
  - all additional $MWh$ are sold @ 14 $/MWh$

Graph showing:
- Non-linear objective function
- $x_H$ and $x_L$ as boundaries
- 50,000 MWh
**EXAMPLE 5: DECISION VARIABLES**

<table>
<thead>
<tr>
<th>Variable</th>
<th>quantity denoted</th>
<th>units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^i_H$</td>
<td>energy sold at 20 $/MWh</td>
<td>MWh</td>
</tr>
<tr>
<td>$x^i_L$</td>
<td>energy sold at 14 $/MWh</td>
<td>MWh</td>
</tr>
<tr>
<td>$w^i_A$</td>
<td>plant A water supply for generation</td>
<td>kAf</td>
</tr>
<tr>
<td>$w^i_B$</td>
<td>plant B water supply for generation</td>
<td>kAf</td>
</tr>
<tr>
<td>$s^i_A$</td>
<td>reservoir A spill</td>
<td>kAf</td>
</tr>
<tr>
<td>$s^i_B$</td>
<td>reservoir B spill</td>
<td>kAf</td>
</tr>
<tr>
<td>$r^i_A$</td>
<td>reservoir A end of period i level</td>
<td>kAf</td>
</tr>
<tr>
<td>$r^i_B$</td>
<td>reservoir B end of period i level</td>
<td>kAf</td>
</tr>
</tbody>
</table>

superscript $i$ denotes period $i$, $i = 1, 2$
EXAMPLE 5: OBJECTIVE FUNCTION

maximize total revenues from sales

$$\max \quad Z = 20(x^1_H + x^2_H) + 14(x^1_L + x^2_L)$$

4 of the 16 decision variables
2 for each period

units of $Z$ are in $\$$
EXAMPLE 5: CONSTRAINTS

- Period 1 constraints

- Energy conservation in a lossless system
  - total generation \( 400 w_A^1 + 200 w_B^1 \) (MWh)
  - total sales \( x_H^1 + x_L^1 \) (MWh)

- Losses are neglected and so
  \[ x_H^1 + x_L^1 = 400 w_A^1 + 200 w_B^1 \]

- Maximum available capacity limits
  \[ w_A^1 \leq 150 \]
  \[ w_B^1 \leq 87.5 \]
EXAMPLE 5: CONSTRAINTS

reservoir conservation of flow relations

1. reservoir A:

\[ w_A^1 + s_A^1 + r_A^1 = 1900 + 200 = 2100 \text{ (kAf)} \]

2. reservoir B:

\[ w_B^1 + s_B^1 + r_B^1 = 850 + 40 + w_A^1 + s_A^1 \text{ (kAf)} \]

res. level at e.o.p. 1

res. level at e.o.p. 0

predicted inflow
EXAMPLE 5: CONSTRAINTS

- limitations on reservoir variables

  - reservoir $A$:
    
    \[ 1,200 \leq r_A^1 \leq 2,000 \]  
    \[(kAf)\]

  - reservoir $B$:
    
    \[ 800 \leq r_B^1 \leq 1,500 \]  
    \[(kAf)\]

- sales constraint

  \[ x_H^1 \leq 50,000 \]  
  \[(kAf)\]
EXAMPLE 5 : CONSTRAINTS

- Period 2 constraints
  - energy conservation in a lossless system
    - total generation: \(400 w_A^2 + 200 w_B^2\) (MWh)
    - total sales: \(x_H^2 + x_L^2\) (MWh)
    - losses are neglected and so
      \[x_H^2 + x_L^2 = 400 w_A^2 + 200 w_B^2\]
  - maximum available capacity limits
    \[w_A^2 \leq 150\]
    \[w_B^2 \leq 87.5\]
EXAMPLE 5: CONSTRAINTS

reservoir conservation of flow relations

- reservoir $A$:

\[ w_A^2 + s_A^2 + r_A^2 = r_A^1 + 130 \quad (kAf) \]

- reservoir $B$:

\[ w_B^2 + s_B^2 + r_B^2 = r_B^1 + 15 + w_A^2 + s_A^2 \quad (kAf) \]
EXAMPLE 5: CONSTRAINTS

- limitations on reservoir variables

  - reservoir $A$:
    \[ 1,200 \leq r_A^2 \leq 2,000 \]  
    \[(kAf)\]

  - reservoir $B$:
    \[ 800 \leq r_B^2 \leq 1,500 \]  
    \[(kAf)\]

- sales constraint

  \[ x_H^2 \leq 50,000 \]  
  \[(kAf)\]
EXAMPLE 5 : PROBLEM STATEMENT

- 16 decision variables:

\[ x^i_H, x^i_L, w^i_A, w^i_B, s^i_A, s^i_B, r^i_A, r^i_B, \quad i = 1, 2 \]

- Objective function:

\[ \max Z = 20(x^1_H + x^2_H) + 14(x^1_L + x^2_L) \]

- Constraints:

  - 20 constraints for the periods 1 and 2
  - non-negativity constraints on all variables
EXAMPLE 6: DISHWASHER AND WASHING MACHINE PROBLEM

- The *Appliance Co.* manufactures dishwashers and washing machines.

- The sales targets for next four quarters are:

<table>
<thead>
<tr>
<th>product</th>
<th>variable</th>
<th>quarter t</th>
</tr>
</thead>
<tbody>
<tr>
<td>dishwasher</td>
<td>$D_t$</td>
<td>2,000</td>
</tr>
<tr>
<td>washing machine</td>
<td>$W_t$</td>
<td>1,200</td>
</tr>
</tbody>
</table>
### EXAMPLE 6: QUARTERLY COST COMPONENTS

<table>
<thead>
<tr>
<th>Cost component</th>
<th>Parameter</th>
<th>Quarter $t$ costs ($/unit)</th>
</tr>
</thead>
<tbody>
<tr>
<td>manufacturing ($/unit)</td>
<td>dishwasher</td>
<td>$c_t$</td>
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<td></td>
<td>washing machine</td>
<td>$v_t$</td>
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<td>storage ($/unit)</td>
<td>dishwasher</td>
<td>$j_t$</td>
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<td>washing machine</td>
<td>$k_t$</td>
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<tr>
<td>Hourly labor ($$/hour)</td>
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<td>$p_t$</td>
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</table>
EXAMPLE 6: CONSTRAINTS

- Each dishwasher (washing machine) requires 1.5 (2) hours of labor.
- The labor hours in each quarter cannot grow or decrease by more than 10%; there are 5,000 h of labor in the quarter preceding the first quarter.
- At the start of the first quarter, there are 750 dishwashers and 50 washing machines in storage.
EXAMPLE 6: THE PROBLEM

How to schedule the production in each of the four quarters so as to minimize the costs while meeting the sales targets?
### EXAMPLE 6: QUARTER $t$ DECISION VARIABLES

<table>
<thead>
<tr>
<th>symbol</th>
<th>variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_t$</td>
<td>number of dishwashers produced</td>
</tr>
<tr>
<td>$w_t$</td>
<td>number of washing machines produced</td>
</tr>
<tr>
<td>$r_t$</td>
<td>final inventory of dishwashers</td>
</tr>
<tr>
<td>$s_t$</td>
<td>final inventory of washing machines</td>
</tr>
<tr>
<td>$h_t$</td>
<td>available labor hours during $Q_t$</td>
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</table>

$t = 1, 2, 3, 4$
EXAMPLE 6: OBJECTIVE FUNCTION

minimize the total costs for the four quarters

\[
\min Z = c_1 d_1 + v_1 w_1 + j_1 r_1 + k_1 s_1 + p_1 h_1 \quad \text{quarter 1}
\]
\[
+ c_2 d_2 + v_2 w_2 + j_2 r_2 + k_2 s_2 + p_2 h_2 \quad \text{quarter 2}
\]
\[
+ c_3 d_3 + v_3 w_3 + j_3 r_3 + k_3 s_3 + p_3 h_3 \quad \text{quarter 3}
\]
\[
+ c_4 d_4 + v_4 w_4 + j_4 r_4 + k_4 s_4 + p_4 h_4 \quad \text{quarter 4}
\]
EXAMPLE 6: CONSTRAINTS

Quarterly flow balance relations:

\[
\begin{align*}
    r_{t-1} + d_t - r_t &= D_t \\
    s_{t-1} + w_t - s_t &= W_t
\end{align*}
\]

\( t = 1, 2, 3, 4 \)
EXAMPLE 6: CONSTRAINTS

- Quarterly labor constraints

\[
\begin{align*}
1.5 d_t + 2 w_t - h_t & \leq 0 \\
0.9 h_{t-1} & \leq h_t \leq 1.1 h_{t-1}
\end{align*}
\]

\[ t = 1, 2, 3, 4 \]

\[ h_0 = 5,000 \]
### Example 6: Problem Statement

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<th>$d_1$</th>
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\[
\begin{align*}
125 & \quad 90 & \quad 5.0 & \quad 6.0 & \quad 130 & \quad 100 & \quad 4.5 & \quad 3.8 & \quad 6.0 & \quad 125 & \quad 95 & \quad 4.5 & \quad 3.8 & \quad 6.8 & \quad 126 & \quad 95 & \quad 4.0 & \quad 3.3 & \quad 6.8 & \quad \text{minimize}
\end{align*}
\]
LINEAR PROGRAMMING PROBLEM

\[ \max (\min) \quad Z = c_1 x_1 + \ldots + c_n x_n \]

\[ \text{s.t.} \]

\[ a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1 \]

\[ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = b_2 \]

\[ \vdots \quad \vdots \]

\[ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n = b_m \]

\[ x_1 \geq 0, \ x_2 \geq 0, \ldots, x_n \geq 0 \]

\[ b_1 \geq 0, \ b_2 \geq 0, \ldots, b_m \geq 0 \]
STANDARD FORM OF LP (SFLP)

\[ \begin{align*} 
\max (\min) \ Z & = \mathbf{c}^T \mathbf{x} \\
A \mathbf{x} & = \mathbf{b} \\
\mathbf{x} & \geq 0
\end{align*} \]

- **coefficient matrix** \( A \in \mathbb{R}^{m \times n} \)
- **decision vector** \( \mathbf{x} \in \mathbb{R}^n \)
- **requirement vector** \( \mathbf{b} \in \mathbb{R}^m \)
- **profits (costs) vector** \( \mathbf{c} \in \mathbb{R}^n \)
CONVERSION OF LP INTO SFLP

- An inequality may be converted into an equality by defining an additional nonnegative slack variable.

\[ x_{\text{slack}} \geq 0 \]

- replace the given inequality \( \leq b \) by

\[ \text{inequality} + x_{\text{slack}} = b \]

- replace the given inequality \( \geq b \) by

\[ \text{inequality} - x_{\text{slack}} = b \]
CONVERSION OF LP INTO SFLP

- An unsigned variable $x_u$ is one whose sign is not specified.
- $x_u$ may be converted into two signed variables $x_+$ and $x_-$ with

$$x_+ = \begin{cases} x_u & x_u \geq 0 \\ 0 & x_u < 0 \end{cases}$$

and

$$x_- = \begin{cases} 0 & x_u \geq 0 \\ -x_u & x_u < 0 \end{cases}$$

so that $x_u$ is replaced by

$$x_u = x_+ - x_-$$
SFLP CHARACTERISTICS

- \( \mathbf{x} \) is feasible if and only if \( \mathbf{x} \geq \mathbf{0} \) and \( A\mathbf{x} = \mathbf{b} \)

- \( S = \{ \mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0} \} \) is the feasible region

- \( S = \emptyset \Rightarrow LP \) is infeasible

- \( \mathbf{x}^* \) is optimal \( \Rightarrow \mathbf{c}^T\mathbf{x}^* \geq \mathbf{c}^T\mathbf{x}, \mathbf{x} \in S \)

- \( \mathbf{x}^* \) may be unique, or may have multiple values

- \( \mathbf{x}^* \) may be unbounded