ECE 307 – Techniques for Engineering Decisions

Lecture 2. Introduction to Linear Programming

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OUTLINE

- The nature of a programming or an optimization problem
- Linear programming (LP): salient characteristics
- The LP problem formulation
- The LP problem solution
- Extensive illustrations with numerical examples

EXAMPLE 1: HIGH/LOW HEEL SHOE CHOICE PROBLEM

- A lady is headed to a party and is trying to find a pair of shoes to wear; the choice is narrowed down to two possible choices:
  - a high heel pair; and
  - a low heel pair
- The high heel shoes look more beautiful but are not as comfortable as the competing pair
- Which pair should she choose?

MODEL FORMULATION

- We first quantify our assessment along the two dimensions of *looks* and *comfort* in a table

aspect	*maximum value*	*assessment*		*weighting factor (%)*
		high heels	*low heels*	
aesthetics	5.0	4.2	3.6	70
comfort	5.0	3.5	4.8	30

- Next, we represent the decision in terms of two decision variables:

MODEL FORMULATION

$$x_H = \begin{cases} 1 & \textit{choose high} \\ 0 & \textit{otherwise} \end{cases} \qquad x_L = \begin{cases} 1 & \textit{choose low} \\ 0 & \textit{otherwise} \end{cases}$$

- **We formulate the objective to be the maximization of the *weighted* assessment**

$$max\{70\ \% * aesthetics + 30\ \% * comfort\}$$

- **We state the objective in terms of the defined decision variables**

$$max\, Z = x_H[(4.2)(0.7) + (3.5)(0.3)] + x_L[(3.6)(0.7) + (4.8)(0.3)]$$

MODEL FORMULATION

- Next, we consider the problem constraints:

  - only one pair of shoes can be selected

  - each decision variable is nonnegative

- We express the constraints in terms of x_H and x_L

$$x_H + x_L = 1$$

$$x_H \geq 0, \; x_L \geq 0$$

PROBLEM STATEMENT SUMMARY

- Decision variables:

$$x_H = \begin{cases} 1 & choose\ high \\ 0 & otherwise \end{cases} \qquad x_L = \begin{cases} 1 & choose\ low \\ 0 & otherwise \end{cases}$$

- Objective function:

$$max\, Z = 3.99\, x_H + 3.96\, x_L$$

- Constraints:

$$x_H + x_L = 1$$

$$x_H \geq 0, \; x_L \geq 0$$

THE OPTIMAL SOLUTION

- We determine the values x_H^* and x_L^* which result in the value of Z^* such that

$$Z^* = Z\left(x_H^*, x_L^*\right) \geq Z\left(x_H, x_L\right) \qquad (\dagger)$$

for all *feasible* $\left(x_H, x_L\right)$

- We call such a solution *an optimal solution*
- A *feasible* solution is one that satisfies all the constraints on the problem
- The *optimal* solution, denoted by $\left(x_H^*, x_L^*\right)$, is selected from all the *feasible* solutions to the problem so as to satisfy $(\dagger)$

SOLUTION APPROACH: EXHAUSTIVE SEARCH

- We enumerate all the feasible solutions: in this problem there are only two alternatives:

$$A: \begin{cases} x_H = 1 \\ x_L = 0 \end{cases} \qquad\qquad B: \begin{cases} x_H = 0 \\ x_L = 1 \end{cases}$$

- We evaluate Z for A and B and compare

$$Z_A = 3.99 \qquad\qquad Z_B = 3.96$$

so that $Z_A > Z_B$ and so A is the optimal choice

- The *optimal* solution is

$$x^*_H = 1 , \quad x^*_L = 0 \quad \text{and} \quad Z^* = 3.99$$

CHARACTERISTICS OF A PROGRAMMING/OPTIMIZATION PROBLEM

- The objective is to select the decision among the various alternatives and therefore requires first the *definition* of the *decision variables*
- We determine the "*best*" decision simply based on the objective function value; to do so we require the *mathematical formulation* of the *objective function*
- The decision must satisfy *each specified constraint* and so we require the *mathematical statement* of the *problem constraints*

CLASSIFICATION OF PROGRAMMING PROBLEMS

The problem statement is characterized by :

- **decision variables**
  - continuous valued
  - integer valued
- **objective function**
  - linear
  - non linear
- **constraints**
  - linear
  - non linear

PROGRAMMING PROBLEM CLASSES

- Linear/nonlinear programming
- Static/dynamic programming
- Integer programming
- Mixed programming

EXAMPLE 2: CONDUCTOR PROBLEM

- A company is producing two types of conductors for *EHV* transmission lines

type	*conductor*	*production capacity* (*unit/day*)	*metal needed* (*tons/unit*)	*profits* (*$/unit*)
1	*ACSR* 84/19	4	1/6	3
2	*ACSR* 18/7	6	1/9	5

- The supply department can provide up to 1 *ton* of metal each day
- We schedule the production so as to *maximize* the profits of the company

PROBLEM ANALYSIS

- Formulation of the objective: to *maximize* the profits of the company
- Means to attain this objective: determine how many units of product 1 and of product 2 to produce each day
- Consideration of all the constraints: the daily production capacity limits, the daily metal supply limit and *common sense* requirements

MODEL CONSTRUCTION

- We define the decision variables to be

$$x_1 = \textit{number of type } 1 \textit{ units produced per day}$$

$$x_2 = \textit{number of type } 2 \textit{ units produced per day}$$

- We define the objective to be

$$Z = \textit{profits } (\$/day)$$

$$= 3x_1 + 5x_2$$

- *Sanity check* for units of the objective function

$$(\$/day) = (\$/unit) \cdot (unit/day)$$

PROBLEM STATEMENT

- **Objective function:**

$$max\, Z = 3\,x_1 + 5\,x_2$$

- **Constraints:**
  - **capacity limits:**

$$x_1 \leq 4 \qquad\qquad x_2 \leq 6$$

  - **metal supply limit:**

$$\frac{x_1}{6} + \frac{x_2}{9} \leq 1$$

  - **common sense requirements:**

$$x_1 \geq 0 \;,\; x_2 \geq 0$$

PROBLEM STATEMENT

$$\max Z = 3x_1 + 5x_2$$

$$s.t.$$

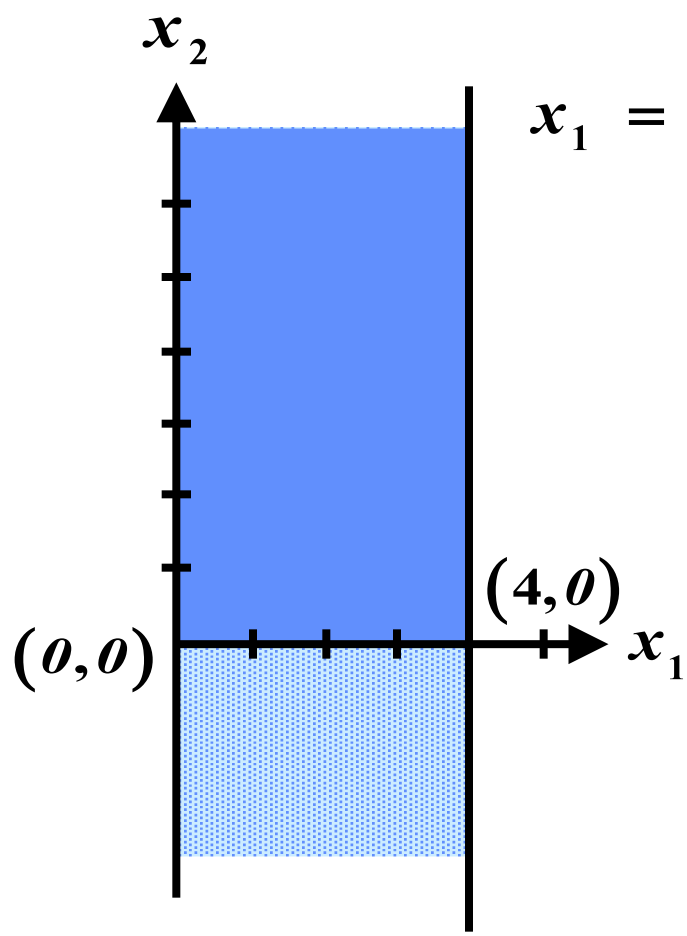
$$x_1 \leq 4$$

$$x_2 \leq 6$$

$$\frac{x_1}{6} + \frac{x_2}{9} \leq 1$$

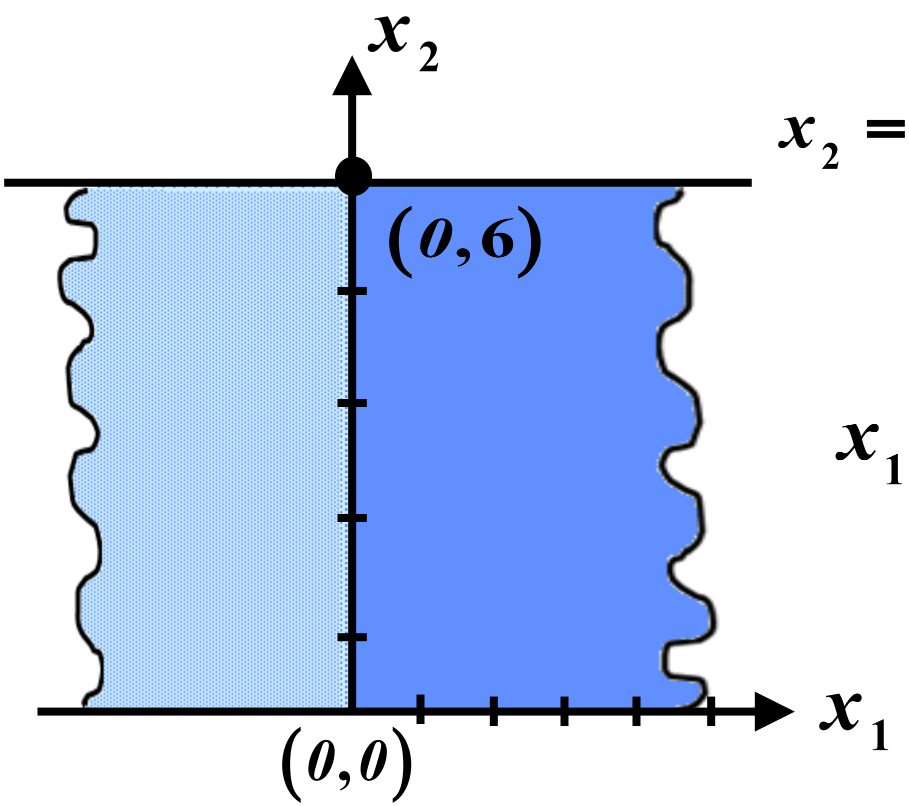
$$x_1 \geq 0 \;,\quad x_2 \geq 0$$

VISUALIZATION OF THE *FEASIBLE REGION*



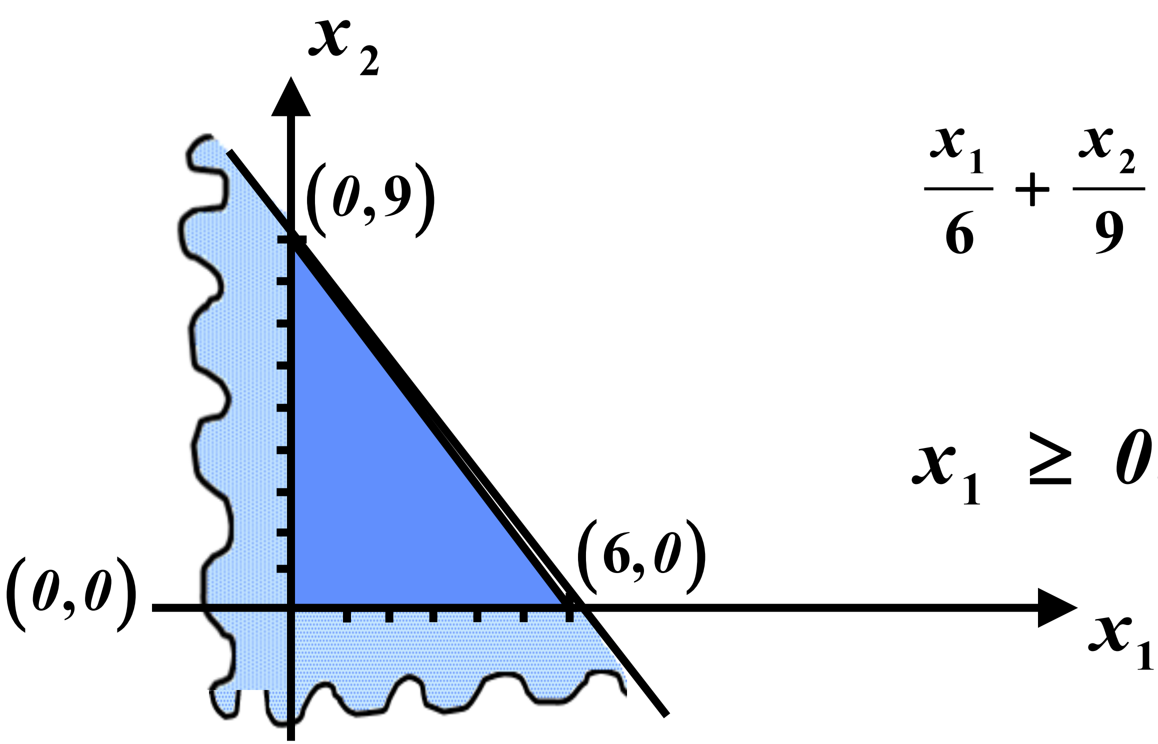
$$x_1 \geq 0 \,,\; x_1 \leq 4 \,,\; x_2 \geq 0$$

VISUALIZATION OF THE *FEASIBLE REGION*



$x_1 \geq 0, \; x_2 \leq 6, \; x_2 \geq 0$

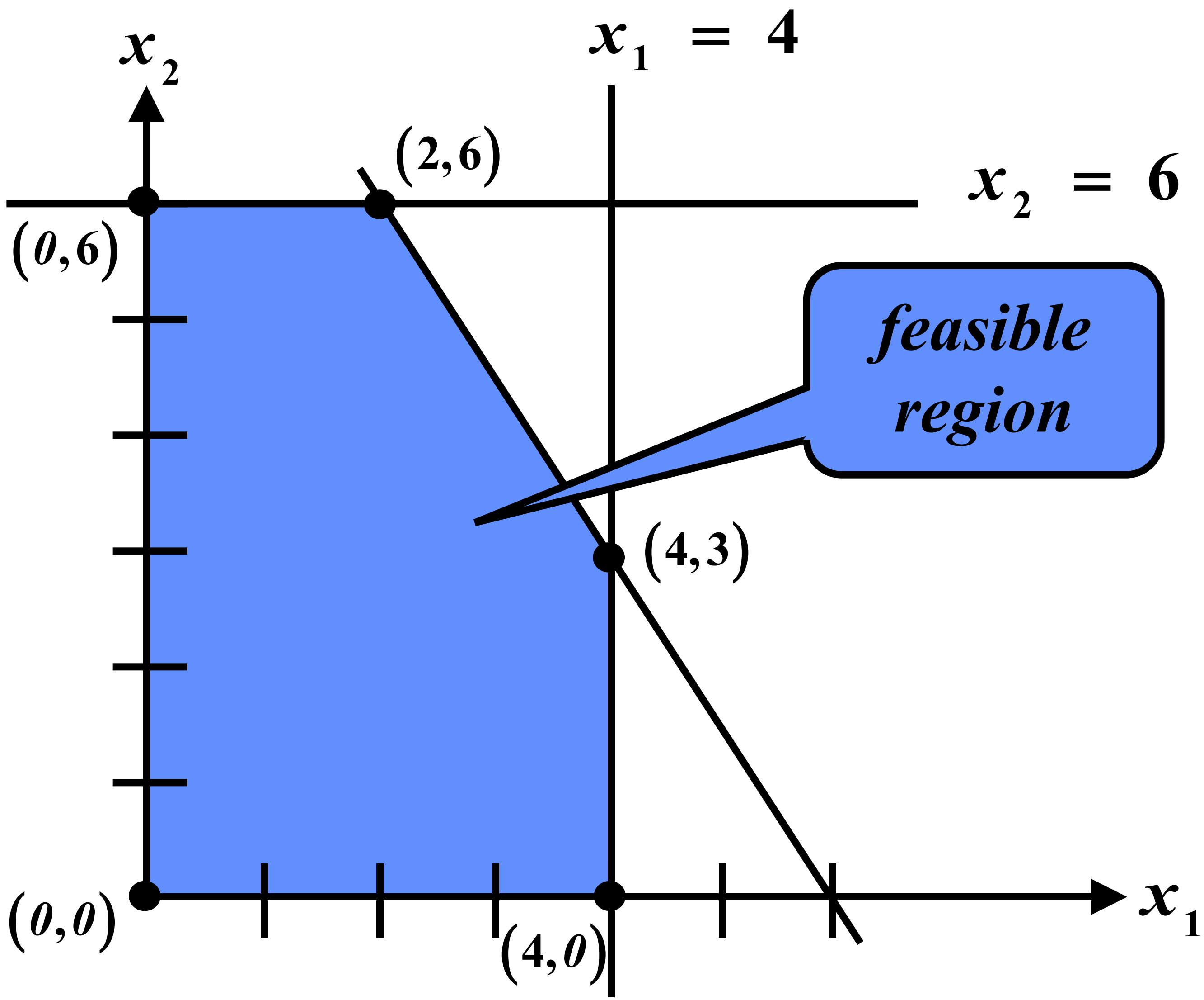
VISUALIZATION OF THE *FEASIBLE REGION*



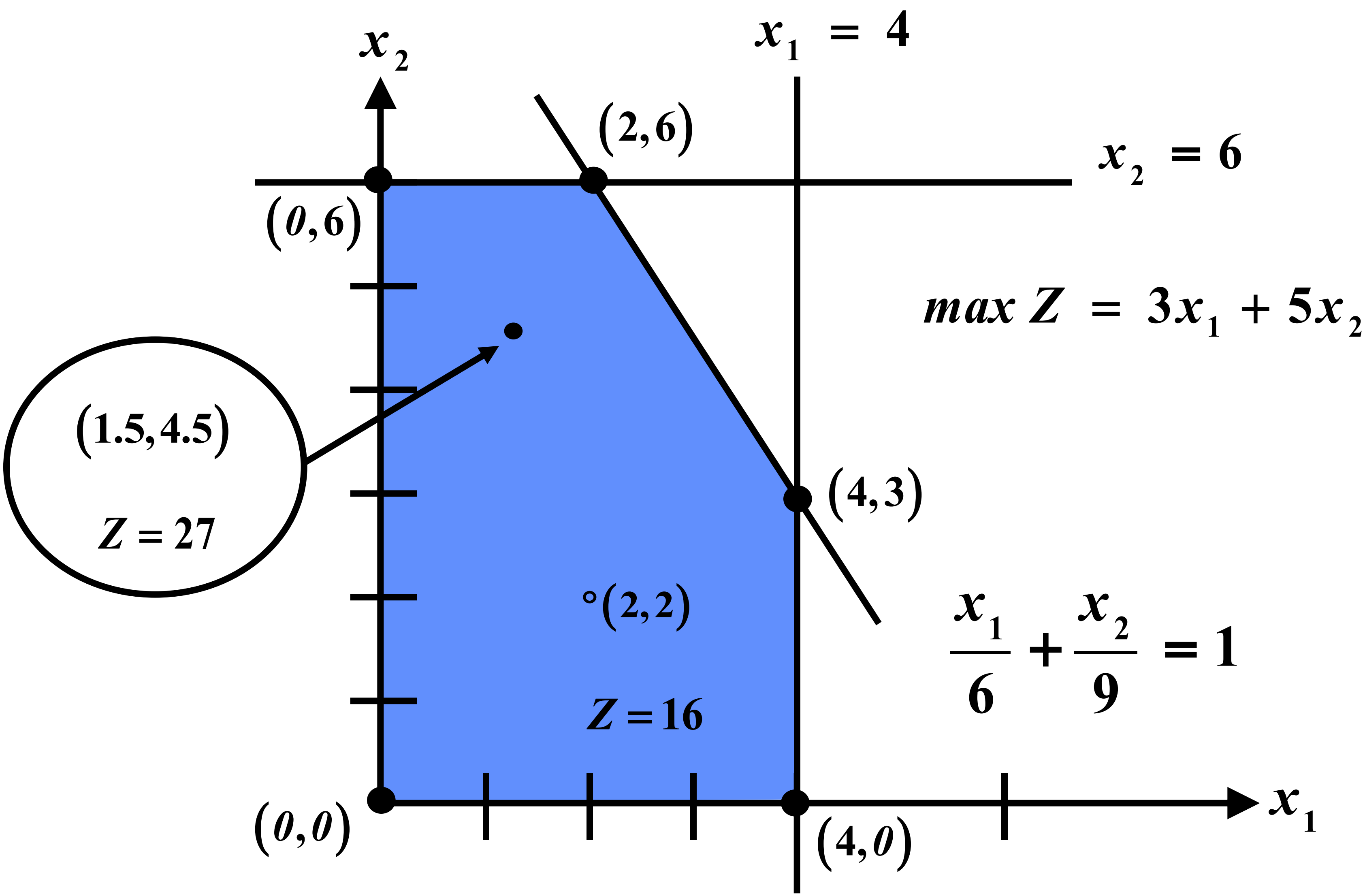
$$\frac{x_1}{6} + \frac{x_2}{9} \leq 1$$

$$x_1 \geq 0,\ x_2 \geq 0$$

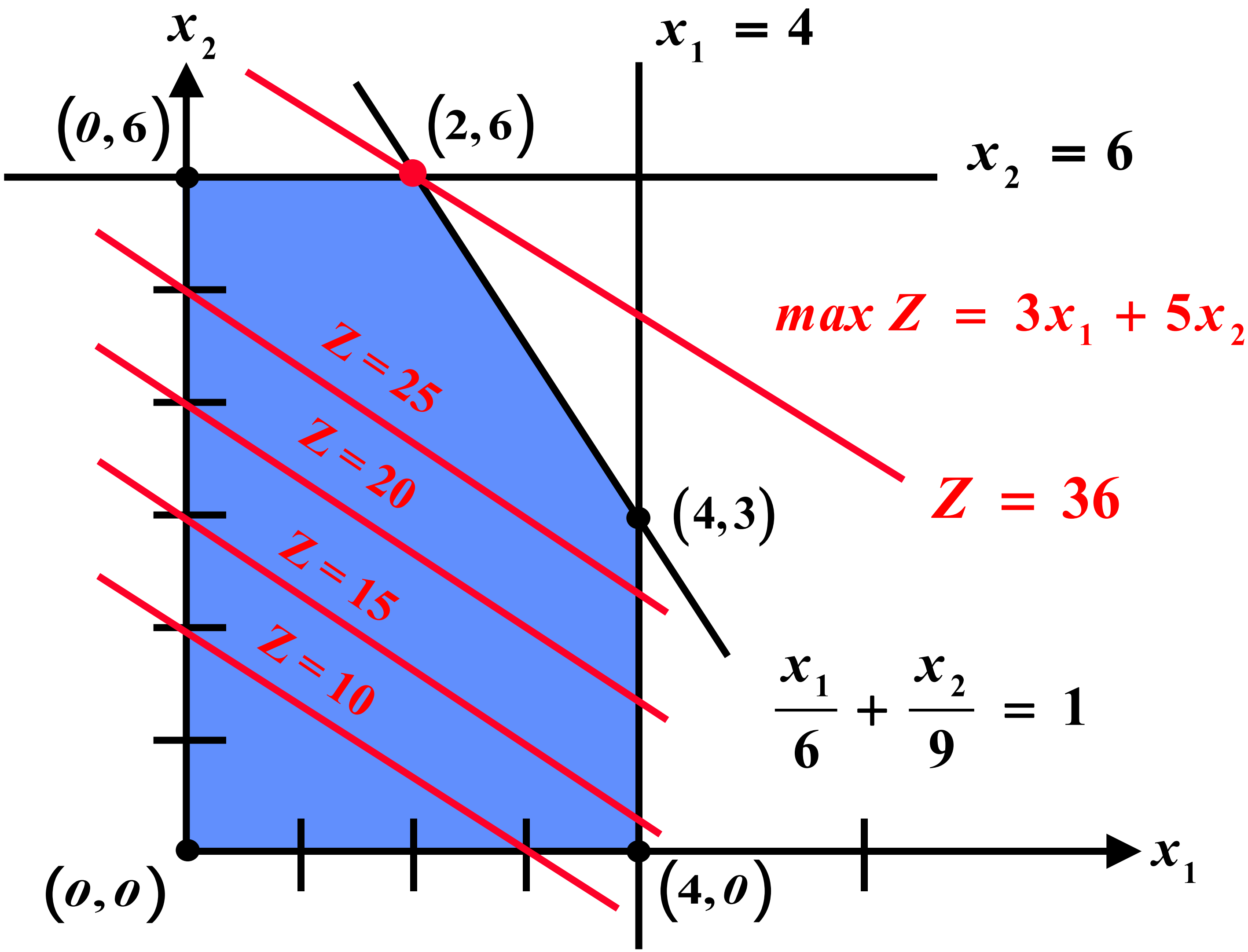
THE *FEASIBLE REGION*



FEASIBLE SOLUTION SPACE



CONTOURS OF CONSTANT Z



OPTIMAL SOLUTION

- For this simple problem, we can *graphically* obtain the *optimal* solution

- The *optimal* solution of this problem is:

$$x_1^* = 2 \quad \text{and} \quad x_2^* = 6$$

- The objective value at the *optimal* solution is

$$Z^* = 3x_1^* + 5x_2^* = 36$$

LINEAR PROGRAMMING (*LP*) PROBLEM DEFINITION

A linear programming problem is an *optimization problem* with a *linear* objective function and *linear* constraints.

EXAMPLE 3: ONE-POTATO, TWO-POTATO PROBLEM

- **Mr. Spud manages the *Potatoes-R-Us Co.* which processes potatoes into packages of freedom fries (F), hash browns (H) and chips (C)**
- **Mr. Spud can buy potatoes from two sources; each source has distinct characteristics/limits**
- **The problem is to determine the respective quantities Mr. Spud needs to buy from source 1 and from source 2 so as to maximize his profits**

EXAMPLE 3: ONE-POTATO, TWO-POTATO PROBLEM

- The given data are summarized in the table

product	*source* 1 *uses* (%)	*source* 2 *uses* (%)	*sales limit* (*tons*)
F	20	30	1.8
H	20	10	1.2
C	30	30	2.4
profits (*$/ton*)	5	6	–

- The following assumptions hold:
  - 30 % waste for each source
  - production may not exceed the sales limit

ANALYSIS

- **Decision variables:**

$$x_1 = \textit{quantity purchased from source } 1$$

$$x_2 = \textit{quantity purchased from source } 2$$

- **Objective function:**

$$max\ Z = 5x_1 + 6x_2$$

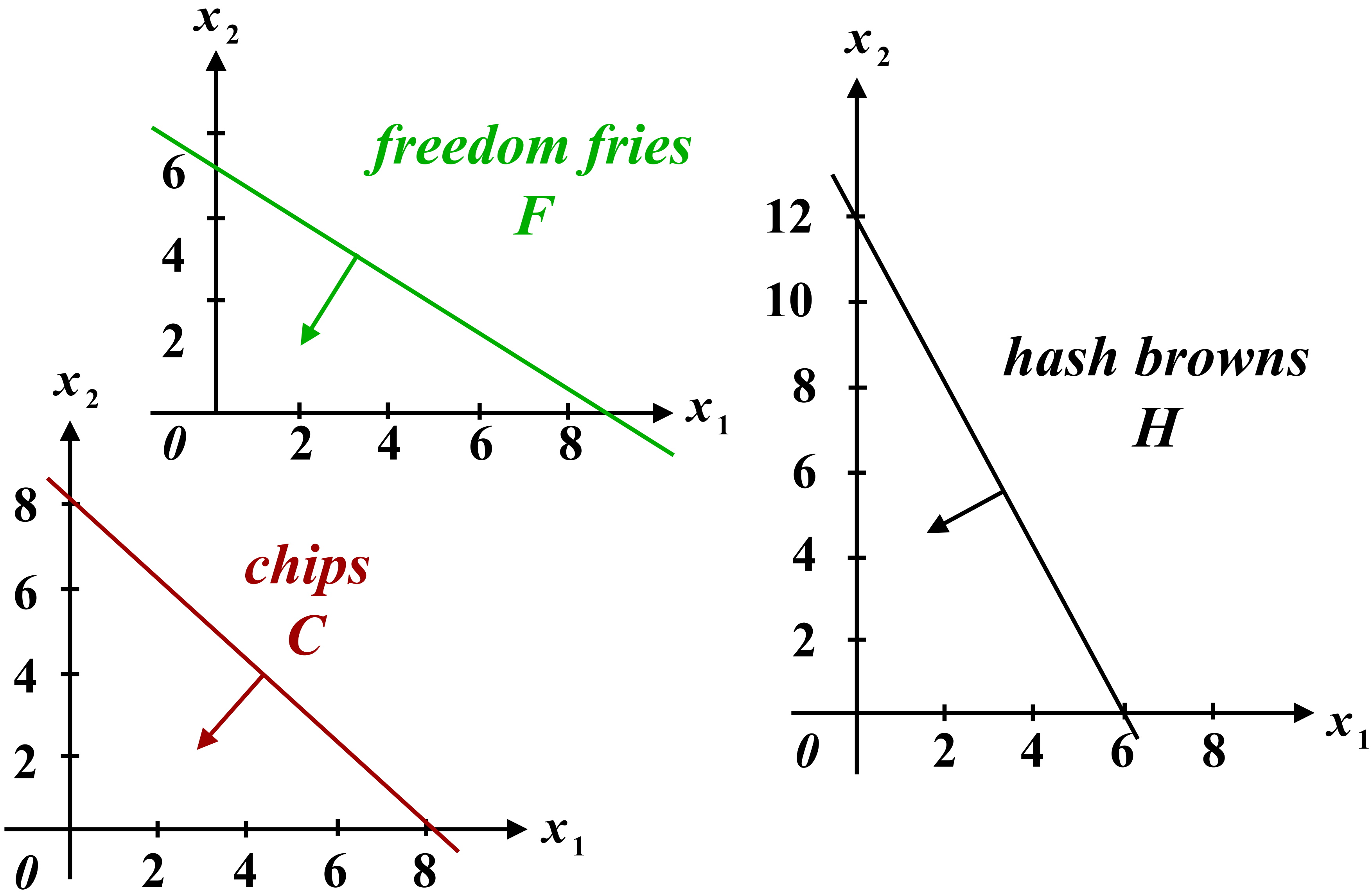
- **Constraints:**

$$0.2\,x_1 + 0.3\,x_2 \le 1.8 \quad (F)$$

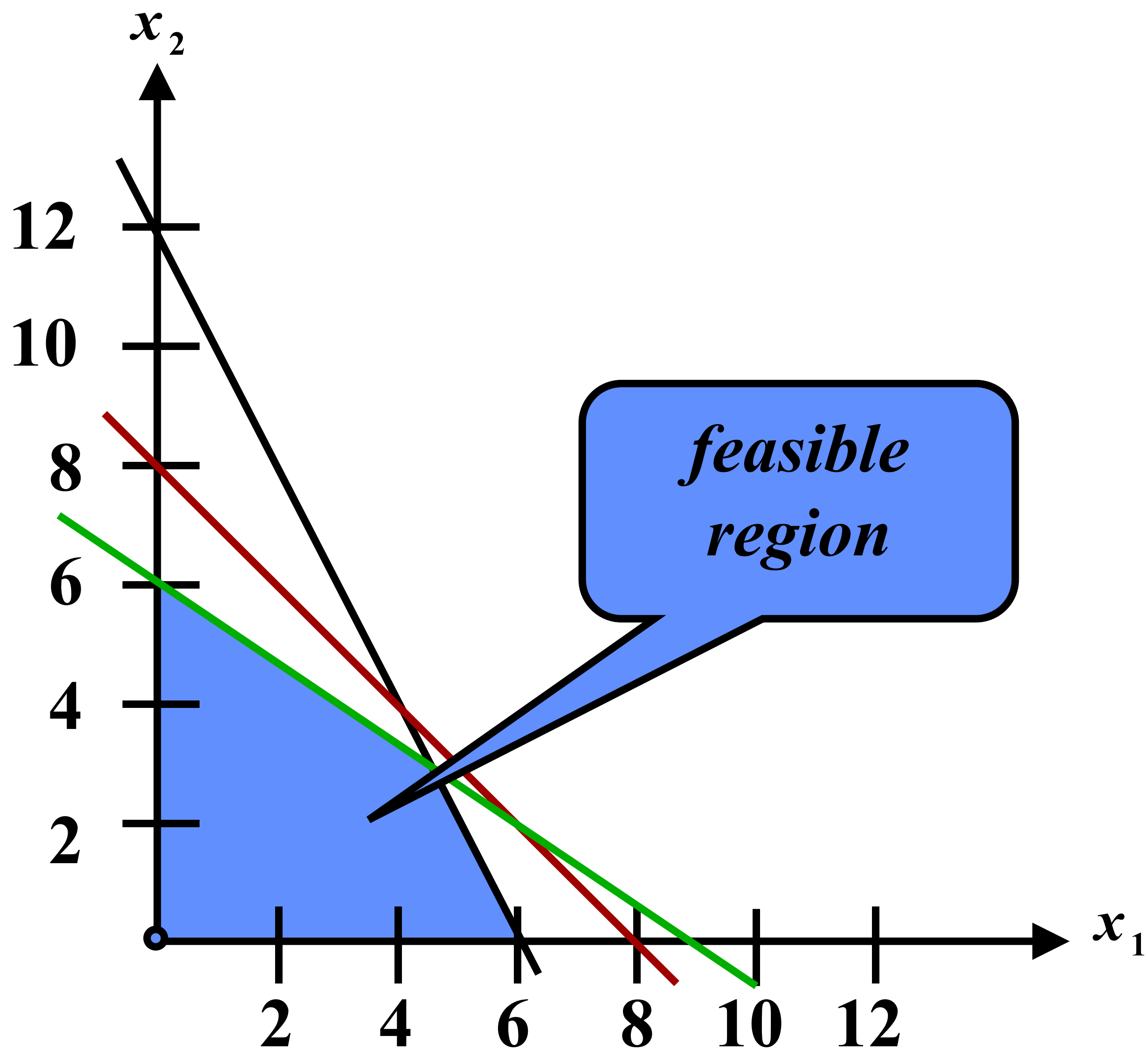
$$0.2\,x_1 + 0.1\,x_2 \le 1.2 \quad (H) \qquad x_1 \ge 0,\ x_2 \ge 0$$

$$0.3\,x_1 + 0.3\,x_2 \le 2.4 \quad (C)$$

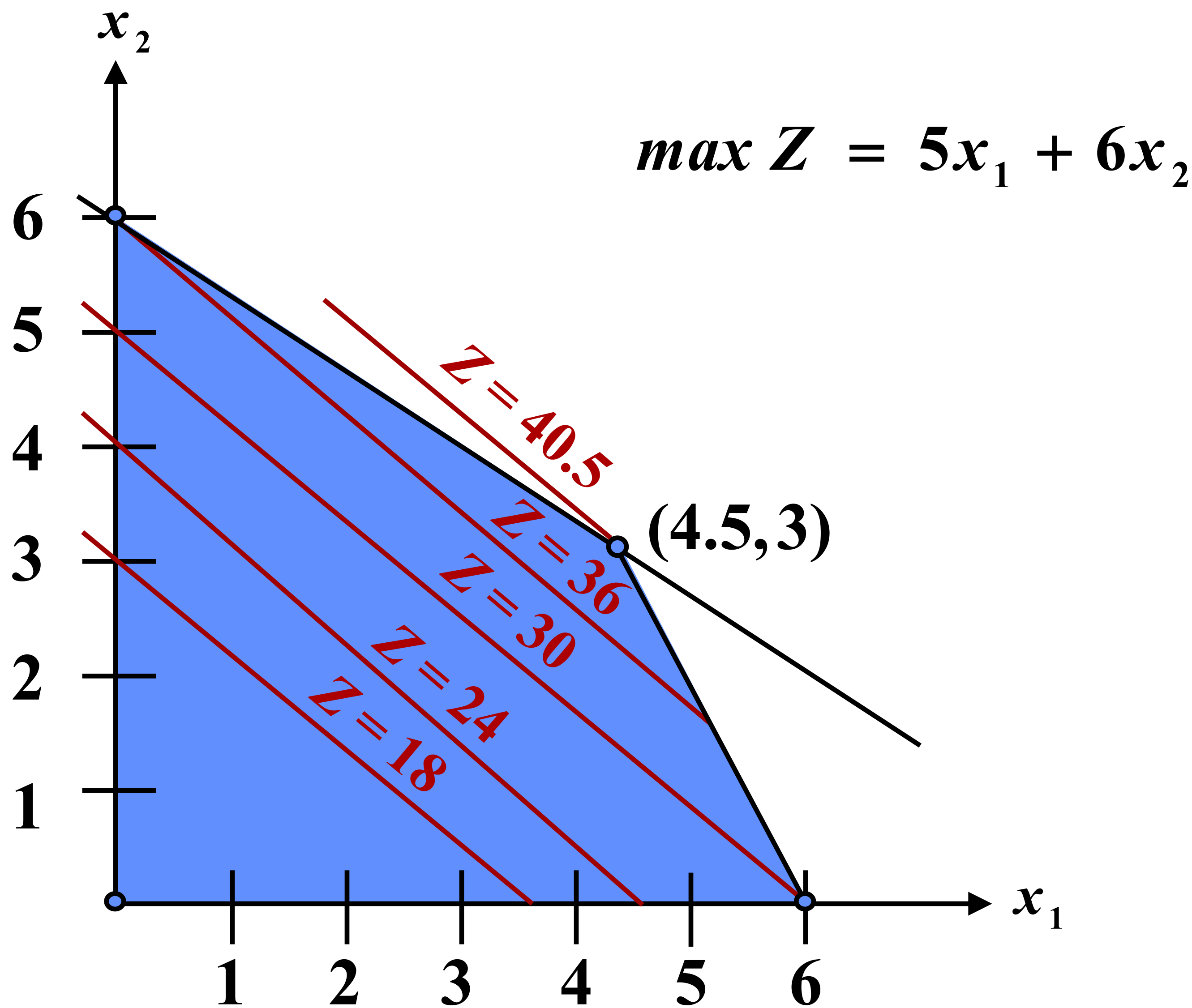
FEASIBLE REGION DETERMINATION



THE FEASIBLE REGION



EXAMPLE 3: CONTOURS OF CONSTANT Z



THE OPTIMAL SOLUTION

- The optimal solution of this problem is:

$$x_1^* = 4.5 \qquad x_2^* = 3$$

- The objective value at the optimal solution is:

$$Z^* = 5x_1^* + 6\,x_2^* = 40.5$$

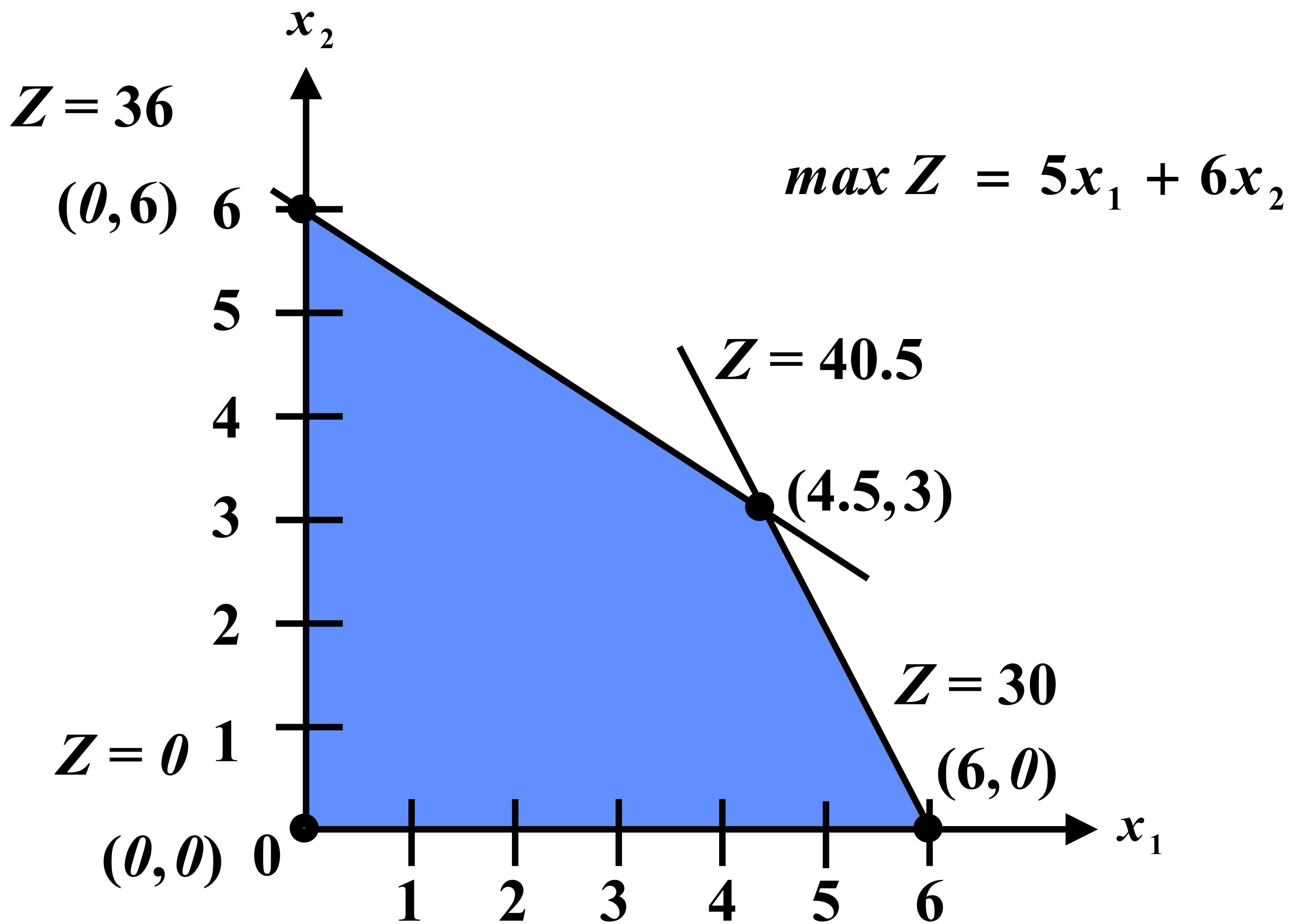
IMPORTANT OBSERVATIONS

- **Constant Z lines are parallel and change monotonically along the direction normal to the contours of constant values of Z**
- **An *optimal* solution must be at one of the *corner points* of the feasible region: fortuitously, there are only a *finite* number of *corner points***
- **If a particular *corner point* gives a better solution (in terms of its Z value) than that at every other adjacent *corner point*, then, it is an *optimal* solution**

CONCEPTUAL SOLUTION PROCEDURE

- Initialization step: start at a *corner point*
- Iteration step: move to an improved *adjacent corner point* and repeat this step as many times as needed
- Stopping rule: stop when the *corner point* solution is better than that at each *adjacent corner point*
- This conceptual procedure forms the basis of the *simplex approach*

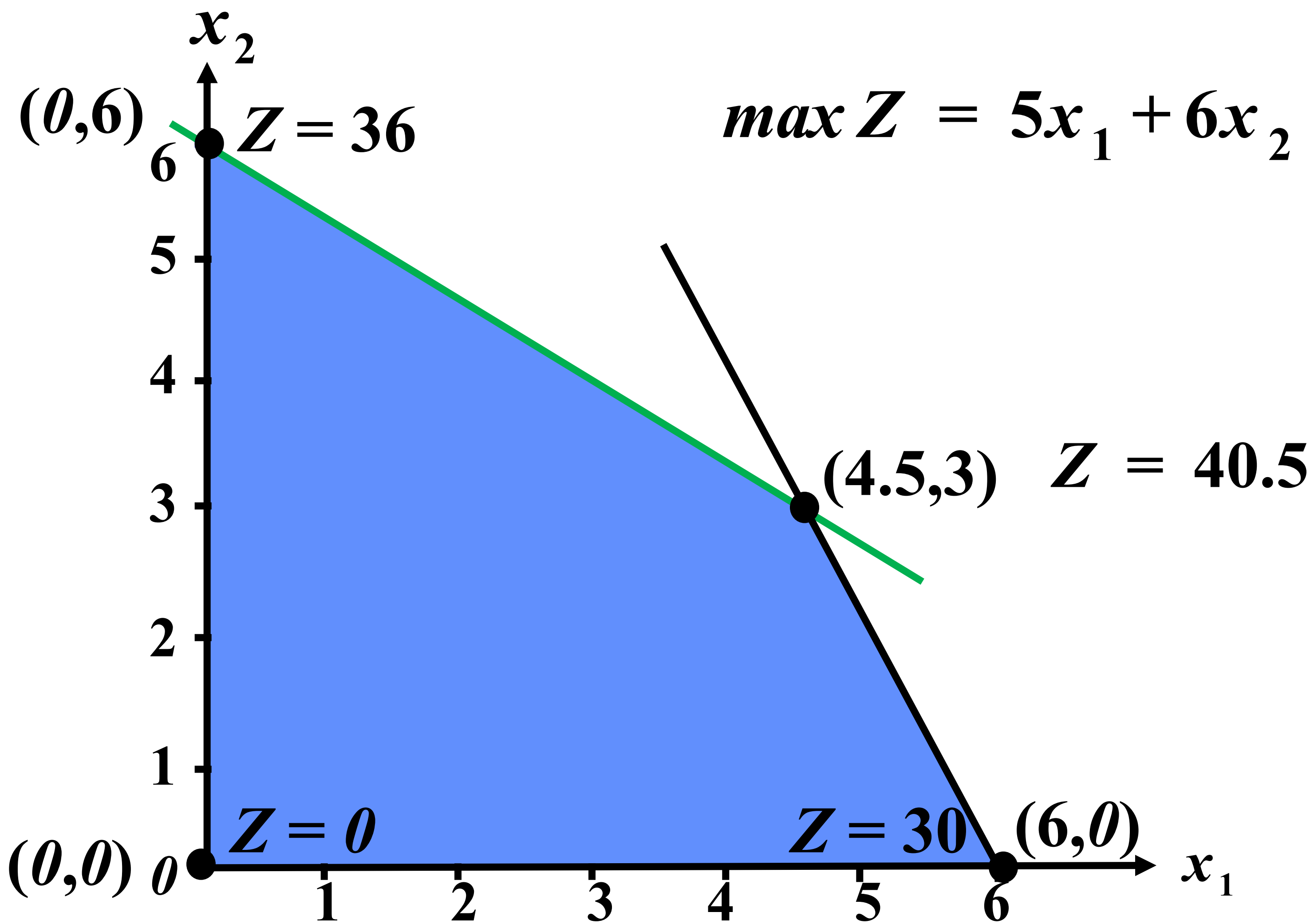
EXAMPLE 3: THE SIMPLEX APPROACH SOLUTION



EXAMPLE 3 : THE SIMPLEX APPROACH SOLUTION

step	x_2	x_1	Z
0	*0*	*0*	*0*
1	0	6	36
2	4.5	3	40.5
3	6	*0*	30

EXAMPLE 3 : THE SIMPLEX APPROACH SOLUTION



EXAMPLE 3 : THE SIMPLEX APPROACH SOLUTION

1. Start at $(0,0)$ with $Z(0,0) = 0$

2. (*i*) Move from $(0,0)$ to $(0,6)$, $Z(0,6) = 36$

   (*ii*) Move from $(0,6)$ to $(4.5,3)$; compute $Z(4.5,3) = 40.5$

3. Compare the objective at $(4.5,3)$ to values at $(6,0)$ and at $(0,6)$:

$$Z(4.5,3) \geq Z(6,0)$$

$$Z(4.5,3) \geq Z(0,6)$$

therefore, $(4.5,3)$ is *optimal*

REVIEW

- **Key requirements of a programming problem:**
  - **to make a decision, we must define the *decision variables***
  - **to achieve the specified objective, we must express mathematically the *objective function***
  - **to ensure *feasibility*, the decision variables must satisfy each *mathematically formulated constraint***

REVIEW

- Key attributes of an *LP*
  - the objective function is *linear*
  - the constraints are *linear*
- Basic steps in formulating a programming problem
  - definition of decision variables
  - statement of the objective function
  - formulation of the constraints

REVIEW

- **Words of caution: care is required with units and attention is needed to not ignore the *implicit constraints*, such as nonnegativity, and the common sense requirements in an *LP* formulation**
- **Graphical solution approach for two–variable problems**
  - **feasible region determination**
  - **contours of constant Z**
  - **identification of the vertex with optimal Z^{*}**

EXAMPLE 4 : QUALITY CONTOL INSPECTION OF GOODS PRODUCED

- There are 8 grade 1 and 10 grade 2 inspectors available for *QC* inspection; at least 1,800 pieces must be inspected in each 8–*hour* day
- Problem data are summarized below:

grade level	*speed* (*unit/h*)	*accuracy* (*%*)	*wages* (*$/h*)
1	25	98	4
2	15	95	3

EXAMPLE 4 : INSPECTION OF GOODS PRODUCED

- **Each error costs *$ 2***
- **The problem is to determine the *optimal* assignment of inspectors, i.e., the number of inspectors of grade 1 and that of grade 2 to result in the least–cost *QC* inspection effort**

EXAMPLE 4 : FORMULATION

- **Definition of decision variables:**

  x_1 = **number of grade 1 inspectors assigned**

  x_2 = **number of grade 2 inspectors assigned**

- **Objective function**

  - **optimal assignment: minimum costs**

  - **costs = wages + errors**

EXAMPLE 4 : FORMULATION

- each grade 1 inspector costs:

$$4 + 2\,(25)(0.02) = 5\ \$/h$$

- each grade 2 inspector costs:

$$3 + 2\,(15)(0.05) = 4.5\ \$/h$$

- total daily inspection costs in $ are

$$Z = 8[5x_1 + 4.5x_2] = 40\,x_1 + 36\,x_2 \qquad (\$)$$

EXAMPLE 4 : FORMULATION

- **Constraints:**
  - **job completion:**

$$\begin{aligned} & 8(25)\,x_1 + 8(15)\,x_2 \geq 1,800 \\ \Leftrightarrow\ & 200\,x_1 + 120\,x_2 \geq 1,800 \\ \Leftrightarrow\ & 5\,x_1 + 3\,x_2 \geq 45 \end{aligned}$$

  - **availability limit:**

$$x_1 \leq 8$$

$$x_2 \leq 10$$

  - **nonnegativity:**

$$x_1 \geq \mathit{0},\ x_2 \geq \mathit{0}$$

EXAMPLE 4 : PROBLEM STATEMENT SUMMARY

- **Decision variables:**

  x_1 = **number of grade 1 inspectors assigned**

  x_2 = **number of grade 2 inspectors assigned**

- **Objective function:**

$$\min Z = 40x_1 + 36x_2$$

- **Constraints:**

$$5x_1 + 3x_2 \geq 45$$

$$x_1 \leq 8$$

$$x_2 \leq 10$$

$$x_1 \geq 0, \; x_2 \geq 0$$

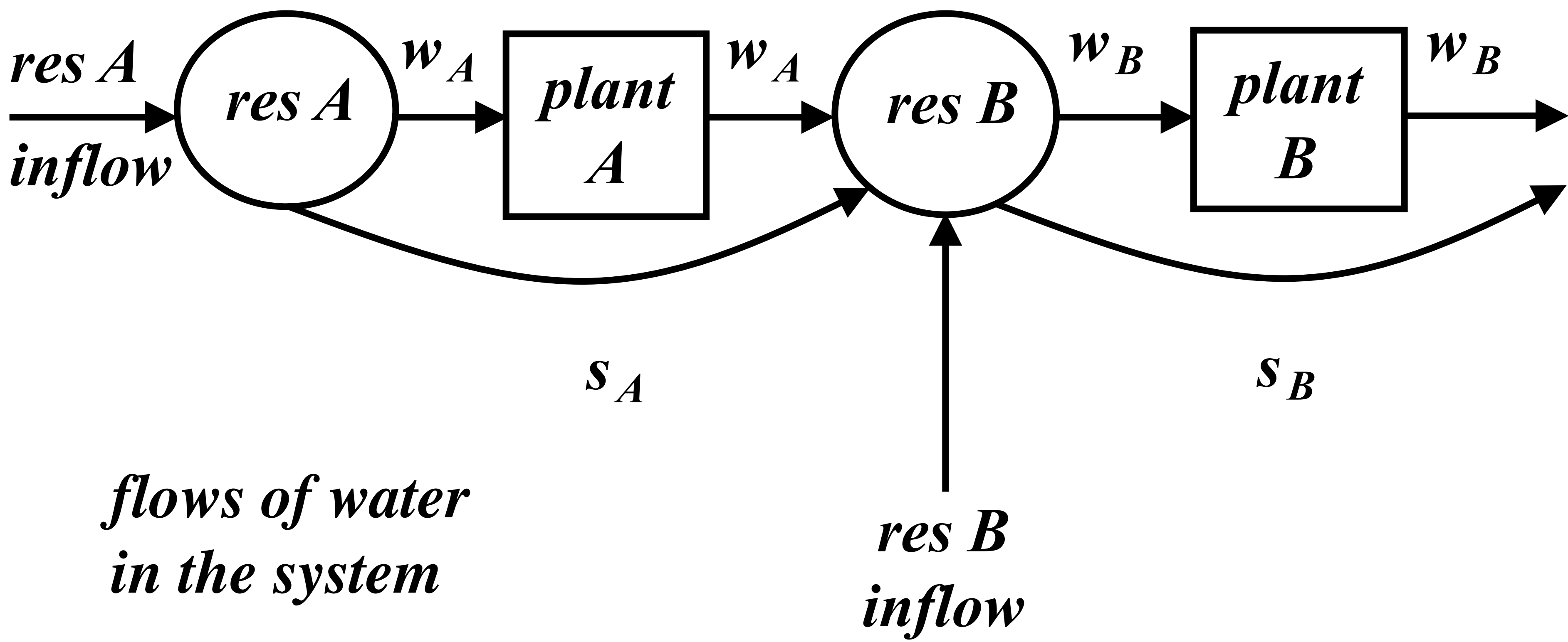
MULTI – PERIOD SCHEDULING

- More than one period is involved
- The result of each period affects the initial conditions for the next period and therefore the solution
- We need to define variables to take into account the initial conditions in addition to the decision variables of the problem

EXAMPLE 5 : HYDROELECTRIC POWER SYSTEM OPERATIONS

- We consider a single operator of a system consisting of two water reservoirs with a hydroelectric plant attached to each reservoir
- We schedule the two power plant operations over a two–period horizon
- We are interested in the plan that maximizes the total revenues of the system operator

EXAMPLE 5 : HYDROELECTRIC POWER SYSTEM OPERATIONS

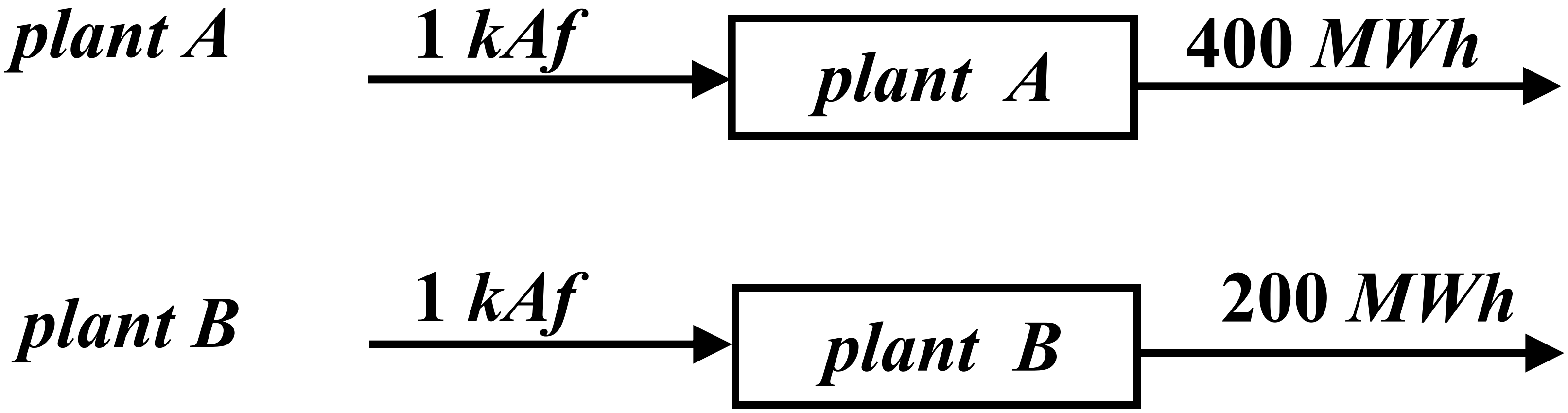


flows of water in the system

EXAMPLE 5 : *kAf* RESERVOIR DATA

parameter	*reservoir A*	*reservoir B*
maximum capacity	2,000	1,500
predicted inflow in period 1	200	40
predicted inflow in period 2	130	15
minimum allowable level	1,200	800
level at start of period 1	1,900	850

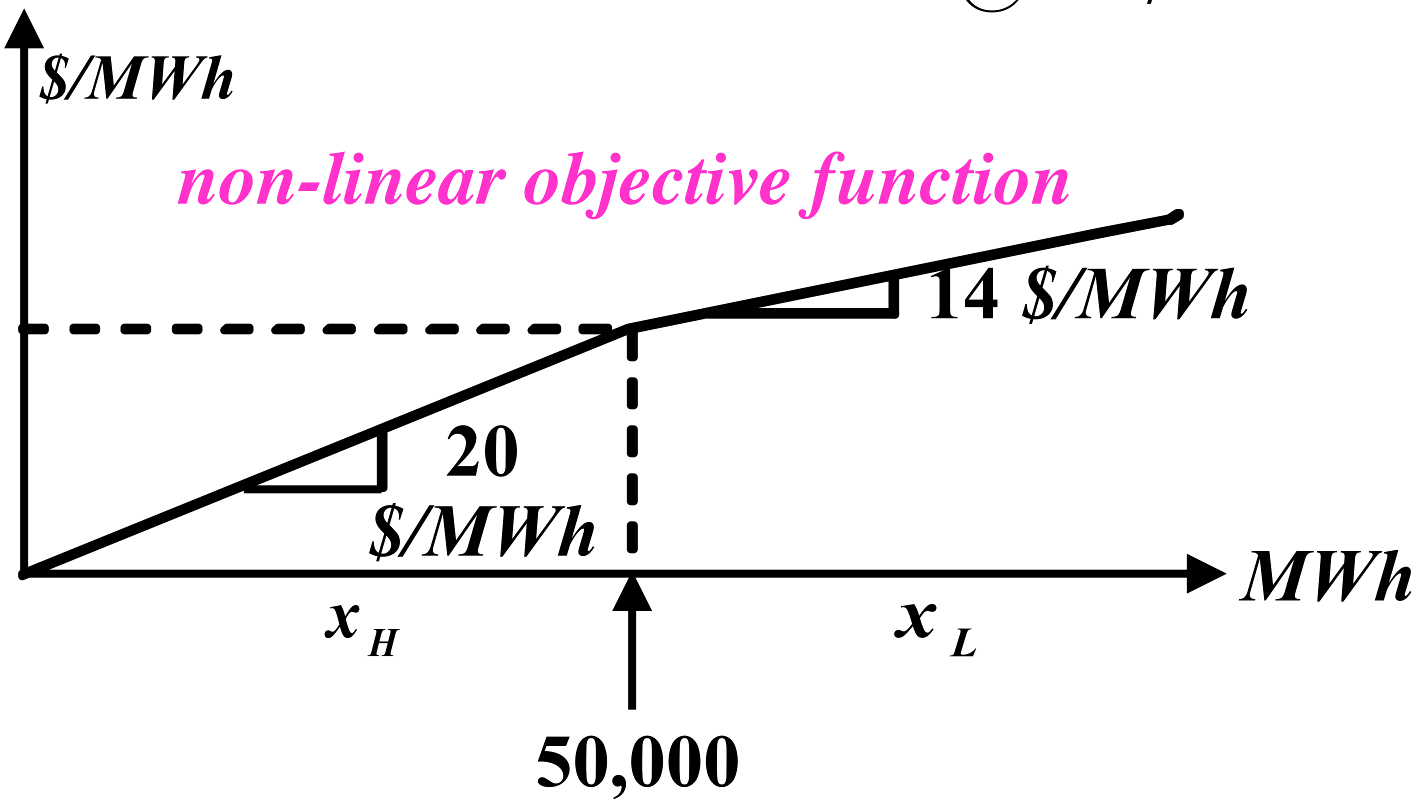
EXAMPLE 5 : SYSTEM CHARACTERISTICS



reservoir	*max kAf for generation per period*
A	150
B	87.5

EXAMPLE 5 : SYSTEM CHARACTERISTICS

- Two–tier price for the *MWh* demand in each period
  - up to 50,000 *MWh* can be sold @ 20 *$ /MWh*
  - all additional *MWh* are sold @ 14 *$ /MWh*



EXAMPLE 5 : DECISION VARIABLES

Variable	*quantity denoted*	*units*
x_H^i	*energy sold at* 20 *$/MWh*	*MWh*
x_L^i	*energy sold at* 14 *$/MWh*	*MWh*
w_A^i	*plant A water supply for generation*	*kAf*
w_B^i	*plant B water supply for generation*	*kAf*
s_A^i	*reservoir A spill*	*kAf*
s_B^i	*reservoir B spill*	*kAf*
r_A^i	*reservoir A end of period i level*	*kAf*
r_B^i	*reservoir B end of period i level*	*kAf*

superscript i **denotes period** $i, i = 1, 2$

EXAMPLE 5 : OBJECTIVE FUNCTION

maximize total revenues from sales

$$max \quad Z = 20(x_H^1 + x_H^2) + 14(x_L^1 + x_L^2)$$

4 of the 16 decision variables

2 for each period

units of Z are in $\$$

EXAMPLE 5 : CONSTRAINTS

- Period 1 constraints
  - energy conservation in a lossless system
    - total generation $400\, w_A^1 + 200\, w_B^1 \quad (MWh)$
    - total sales $x_H^1 + x_L^1 \quad (MWh)$
    - losses are negelected and so

$$x_H^1 + x_L^1 = 400\, w_A^1 + 200\, w_B^1$$

  - maximum available capacity limits

$$w_A^1 \leq 150$$

$$w_B^1 \leq 87.5$$

EXAMPLE 5 : CONSTRAINTS

- reservoir conservation of flow relations

  - reservoir A:

$$w_A^1 + s_A^1 + r_A^1 = 1{,}900 + 200 = 2{,}100\,(kAf)$$

res. level at e.o.p. 1

res. level at e.o.p. 0

predicted inflow

  - reservoir B:

$$w_B^1 + s_B^1 + r_B^1 = 850 + 40 + w_A^1 + s_A^1\,(kAf)$$

EXAMPLE 5 : CONSTRAINTS

- **limitations on reservoir variables**
  - **reservoir *A*:**

$$1,200 \leq r_A^1 \leq 2,000 \qquad (kAf)$$

  - **reservoir *B*:**

$$800 \leq r_B^1 \leq 1,500 \qquad (kAf)$$

- **sales constraint**

$$x_H^1 \leq 50,000 \qquad (kAf)$$

EXAMPLE 5 : CONSTRAINTS

- **Period 2 constraints**
  - **energy conservation in a lossless system**
    - **total generation** $400 w_A^2 + 200 w_B^2 \quad (MWh)$
    - **total sales** $x_H^2 + x_L^2 \quad (MWh)$
    - **losses are neglected and so**

$$x_H^2 + x_L^2 = 400 w_A^2 + 200 w_B^2$$

  - **maximum available capacity limits**

$$w_A^2 \leq 150$$

$$w_B^2 \leq 87.5$$

EXAMPLE 5 : CONSTRAINTS

- reservoir conservation of flow relations

  - reservoir A:

$$w_A^2 + s_A^2 + r_A^2 = r_A^1 + 130 \qquad (kAf)$$

res. level at e.o.p. 2

res. level at e.o.p. 1

predicted inflow

  - reservoir B:

$$w_B^2 + s_B^2 + r_B^2 = r_B^1 + 15 + w_A^2 + s_A^2 \quad (kAf)$$

EXAMPLE 5 : CONSTRAINTS

- limitations on reservoir variables
  - reservoir A:

$$1{,}200 \leq r_A^2 \leq 2{,}000 \qquad (kAf)$$

  - reservoir B:

$$800 \leq r_B^2 \leq 1{,}500 \qquad (kAf)$$

- sales constraint

$$x_H^2 \leq 50{,}000 \qquad (kAf)$$

EXAMPLE 5 : PROBLEM STATEMENT

- **16 decision variables:**

$$x_H^i,\ x_L^i,\ w_A^i,\ w_B^i,\ s_A^i,\ s_B^i,\ r_A^i,\ r_B^i, \qquad i = 1,2$$

- **Objective function:**

$$max \quad Z = 20(x_H^1 + x_H^2) + 14(x_L^1 + x_L^2)$$

- **Constraints:**

  - **20 constraints for the periods 1 and 2**

  - **non-negativity constraints on all variables**

EXAMPLE 6 : DISHWASHER AND WASHING MACHINE PROBLEM

- The *Appliance Co.* manufactures dishwashers and washing machines
- The sales targets for next four quarters are:

product	variable	quarter t			
		1	2	3	4
dishwasher	D_t	2,000	1,300	3,000	1,000
washing machine	W_t	1,200	1,500	1,000	1,400

EXAMPLE 6 : QUARTERLY COST COMPONENTS

cost component		*parameter*	*quarter t costs ($/unit)*			
			1	2	3	4
manufacturing ($/unit)	*dishwasher*	c_t	125	130	125	126
	washing machine	v_t	90	100	95	95
storage ($/unit)	*dishwasher*	j_t	5.0	4.5	4.5	4.0
	washing machine	k_t	4.3	3.8	3.8	3.3
hourly labor ($ /hour)		p_t	6.0	6.0	6.8	6.8

EXAMPLE 6 : CONSTRAINTS

- Each dishwasher (washing machine) requires 1.5 (2) *hours* of labor
- The labor hours in each quarter cannot grow or decrease by more than 10 %; there are 5,000 *h* of labor in the quarter preceding the first quarter
- At the start of the first quarter, there are 750 dish-washers and 50 washing machines in storage

EXAMPLE 6 : THE PROBLEM

How to schedule the production in each of the four quarters so as to minimize the costs while meeting the sales targets?

EXAMPLE 6 : QUARTER t DECISION VARIABLES

symbol	*variable*
d_t	*number of dishwashers produced*
w_t	*number of washing machines produced*
r_t	*final inventory of dishwashers*
s_t	*final inventory of washing machines*
h_t	*available labor hours during Q_t*
$t = 1, 2, 3, 4$	

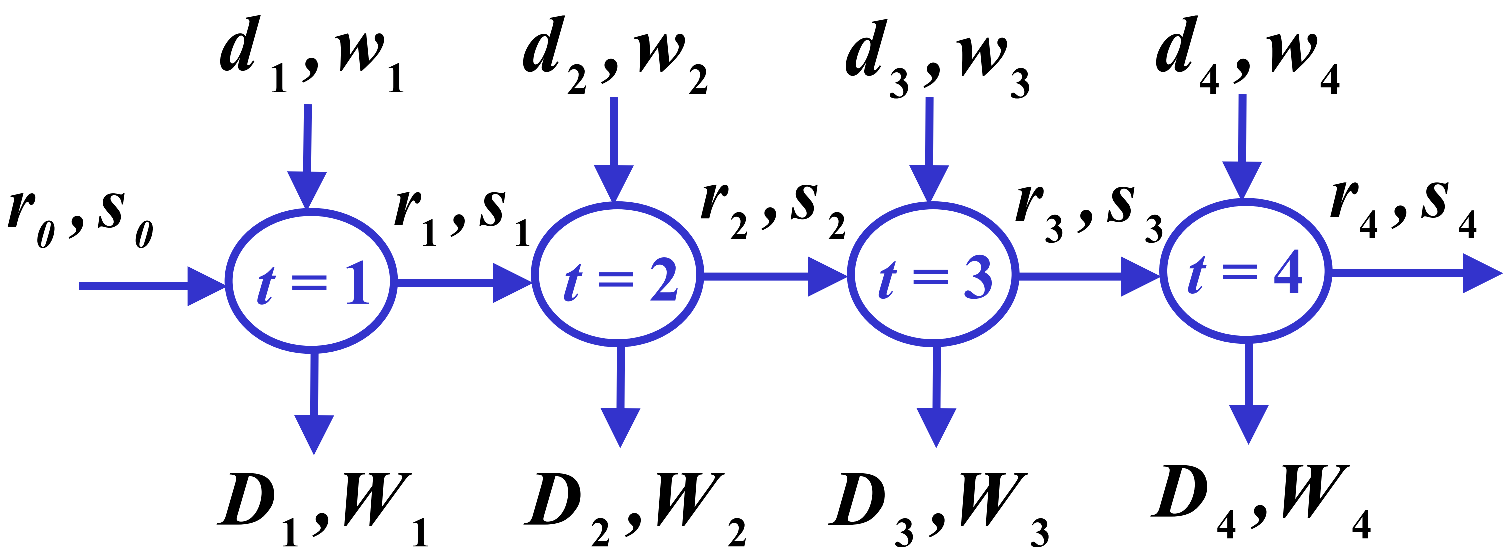
EXAMPLE 6 : OBJECTIVE FUNCTION

minimize the *total* costs for the four quarters

$$
\begin{aligned}
\min Z \;=\; & \overbrace{c_1 d_1 + v_1 w_1}^{\textit{manufacturing costs}} + \overbrace{j_1 r_1 + k_1 s_1}^{\textit{storage costs}} + \overbrace{p_1 h_1}^{\textit{labor costs}} \longleftarrow \textit{quarter } 1 \\
& + c_2 d_2 + v_2 w_2 + j_2 r_2 + k_2 s_2 + p_2 h_2 \longleftarrow \textit{quarter } 2 \\
& + c_3 d_3 + v_3 w_3 + j_3 r_3 + k_3 s_3 + p_3 h_3 \longleftarrow \textit{quarter } 3 \\
& + c_4 d_4 + v_4 w_4 + j_4 r_4 + k_4 s_4 + p_4 h_4 \longleftarrow \textit{quarter } 4
\end{aligned}
$$

EXAMPLE 6 : CONSTRAINTS

- **Quarterly flow balance relations:**



$$\begin{cases} r_{t-1} + d_t - r_t = D_t \\ s_{t-1} + w_t - s_t = W_t \end{cases} \qquad t = 1, 2, 3, 4$$

EXAMPLE 6 : CONSTRAINTS

- Quarterly labor constraints

$$\begin{cases} 1.5\,d_t + 2\,w_t - h_t \le 0 \\ \\ 0.9\,h_{t-1} \le h_t \le 1.1 h_{t-1} \end{cases} \qquad t = 1, 2, 3, 4$$

$$h_0 = 5,000$$

EXAMPLE 6 : PROBLEM STATEMENT

d_1	w_1	r_1	s_1	h_1	d_2	w_2	r_2	s_2	h_2	d_3	w_3	r_3	s_3	h_3	d_4	w_4	r_4	s_4	h_4	
1		-1																		$=$ 1250
	1		-1																	$=$ 1150
1.5	2			-1																$\leq$ 0
				1																$\geq$ 4500
				1																$\leq$ 5500
		1			1		-1													$=$ 1300
			1			1		-1												$=$ 1500
					1.5	2			-1											$\leq$ 0
				-0.9					1											$\geq$ 0
				-1.1					1											$\leq$ 0
							1			1		-1								$=$ 3000
								1			1		-1							$=$ 1000
										1.5	2			-1						$\leq$ 0
									-0.9					1						$\geq$ 0
									-1.1					1						$\leq$ 0
												1			1		-1			$=$ 1000
													1			1		-1		$=$ 1400
															1.5	2			-1	$\leq$ 0
														-0.9					1	$\geq$ 0
														-1.1					1	$\leq$ 0
125	90	5.0	4.3	6.0	130	100	4.5	3.8	6.0	125	95	4.5	3.8	6.8	126	95	4.0	3.3	6.8	*minimize*

LINEAR PROGRAMMING PROBLEM

$$\max\ (\min) \quad Z = c_1 x_1 + \ldots + c_n x_n$$

s.t.

$$\begin{array}{ccccccc}
a_{11} x_1 & + & a_{12} x_2 & + & \ldots + & a_{1n} x_n & = b_1 \\
a_{21} x_1 & + & a_{22} x_2 & + & \ldots + & a_{2n} x_n & = b_2 \\
\vdots & & & & & \vdots & \\
a_{m1} x_1 & + & a_{m2} x_2 & + & \ldots + & a_{mn} x_n & = b_m
\end{array}$$

$$x_1 \geq 0,\ x_2 \geq 0, \ldots , x_n \geq 0$$

$$b_1 \geq 0, b_2 \geq 0, \ldots , b_m \geq 0$$

STANDARD FORM OF *LP* (*SFLP*)

$$\max(\min)\, Z = \underline{c}^{T}\underline{x}$$

$$\underline{A}\,\underline{x} = \underline{b}$$

$$\underline{x} \geq \underline{0}$$

coefficient matrix

$\underline{A} \in \mathbb{R}^{m \times n}$

decision vector

$\underline{x} \in \mathbb{R}^{n}$

$\underline{b} \in \mathbb{R}^{m}$

$\underline{c} \in \mathbb{R}^{n}$

requirement vector

profits (costs) vector

CONVERSION OF *LP* INTO *SFLP*

- An inequality may be converted into an equality by defining an additional *nonnegative* *slack* variable

  $x_{slack} \geq 0$

  - replace the given *inequality* $\leq b$ by

    $$inequality + x_{slack} = b$$

  - replace the given *inequality* $\geq b$ by

    $$inequality - x_{slack} = b$$

CONVERSION OF *LP* INTO *SFLP*

- An unsigned variable x_u is one whose sign is *not* specified
- x_u may be converted into two signed variables x_+ and x_- with

$$x_+ = \begin{cases} x_u & x_u \geq 0 \\ 0 & x_u < 0 \end{cases} \qquad x_- = \begin{cases} 0 & x_u \geq 0 \\ -x_u & x_u < 0 \end{cases}$$

so that x_u is replaced by

$$x_u = x_+ - x_-$$

SFLP CHARACTERISTICS

- $\underline{x}$ is feasible if and only if $\underline{x} \geq \underline{0}$ and $\underline{A}\,\underline{x} = b$
- $S = \{\underline{x} \mid \underline{A}\,\underline{x} = \underline{b},\ \underline{x} \geq \underline{0}\}$ is the feasible region
- $S = \varnothing \Rightarrow LP$ is infeasible
- $\underline{x}^*$ is optimal $\Rightarrow\ \underline{c}^T \underline{x}^* \geq \underline{c}^T \underline{x}\ ,\ \underline{x} \in S$
- $\underline{x}^*$ may be unique, or may have multiple values
- $\underline{x}^*$ may be unbounded

