ECE 307 – Techniques for Engineering Decisions

12. Probability Distributions

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SCOPE OF LECTURE

- We review basic probability distributions
- The entire lecture is simply a review of known probability material given the course prerequisites
- We extensively rely on examples to drive home the usefulness of the material
- We rely on the use of probability distribution tables, as found in the appendices of the Clemen book
OUTLINE OF DISTRIBUTION REVIEWED

- Discrete distributions
  - binomial
  - Poisson

- Continuous distributions
  - exponential
  - normal
Binomial distributions are used to describe events with only two possible outcomes.

Basic requirements are:
- *dichotomous outcomes*: uncertain events occur in a sequence with each event having one of two possible outcomes such as:
THE BINOMIAL DISTRIBUTION

- success/failure
- correct/incorrect
- on/off
- true/false

- constant probability: each event has the same probability of success
- independence: the outcome of each event is independent of the outcomes of any other event
We consider a group of $n$ identical machines with each machine having one of two states:

$$P \{ \text{machine is on} \} = p$$

$$P \{ \text{machine is off} \} = q = 1 - p$$

For concreteness, let us set $n = 8$ and define for $i = 1, 2, \ldots, 8$, the r.v. $s:$
BINOMIAL DISTRIBUTION EXAMPLE

\[ X_i = \begin{cases} 
1 & \text{machine } i \text{ is on with prob. } p \\
0 & \text{machine } i \text{ is off with prob. } q = 1 - p 
\end{cases} \]

- The probability that 3 or more machines are on is determined by the evaluation of the probability

\[ P \left\{ \sum_{i=1}^{8} X_i \geq 3 \right\} = P \left\{ 3 \text{ or more machines are on} \right\} \]
BINOMIAL DISTRIBUTION EXAMPLE

\[ = P \left\{ 3 \text{ machines are on} \right\} + P \left\{ 4 \text{ machines are on} \right\} + \ldots + P \left\{ 8 \text{ machines are on} \right\} \]

\[ P \left\{ \sum_{i=1}^{8} X_i \geq 3 \right\} = \sum_{r=3}^{8} \frac{8!}{(8-r)!r!} p^r (1-p)^{8-r} \]
In general, for a r.v. $\sim$ with dichotomous outcomes of success and failure, the probability of $r$ successes in $n$ trials is

$$P\left\{ R = r \text{ in } n \text{ trials with probability of success } p \right\}$$

$$= \frac{n!}{(n-r)! r!} p^r (1-p)^{n-r}$$

the binomial distribution
We can show that:

\[
E \{ R \} = np
\]

\[
\text{var} \{ R \} = np(1-p)
\]

\[
P \left\{ \sum_{i=1}^{n} X_i \geq k \right\} = \sum_{r=k}^{n} \frac{n!}{(n-r)! \ r!} p^r (1-p)^{n-r}
\]
EXAMPLE: SOFT PRETZELS

- A pretzel entrepreneur can sell each pretzel at $0.50 with a market potential of 100,000 pretzels within a year; as there exists a competing product, he is not be the only seller.

- Basic model is binomial:
  
  - new pretzel is a hit \( \iff \) captures 30% of market in one year
  
  (success)

  - new pretzel is a flop \( \iff \) captures 10% of market in one year

  (failure)
EXAMPLE: SOFT PRETZELS

- The probability of these two outcomes is equal

- Market tests are conducted with 20 pretzels taste tested against the competition; the result indicates that 5 out of 20 testers prefer the new pretzel

- We evaluate the conditional probability

\[
P\left\{\text{new pretzel is a hit} \mid 5 \text{ out of 20 people prefer new pretzel}\right\}
\]
EXAMPLE: SOFT PRETZELS

- We define the success r.v.

\[ S = \begin{cases} 1 & \text{new pretzel is a hit (success)} \\ 0 & \text{otherwise (failure)} \end{cases} \]

with

\[ P\{S = 1\} = P\{S = 0\} = 0.5 \]

and

\[ X_i = \begin{cases} 1 & \text{person } i \text{ prefers new pretzel} \\ 0 & \text{otherwise} \end{cases} \]

- We evaluate

\[ P\{\text{new pretzel is a hit} \mid \text{5 out of 20 people prefer new pretzel}\} \]
EXAMPLE: SOFT PRETZELS

\[ P \left\{ S = 1 \left| \sum_{i=1}^{20} X_i = 5 \right. \right\} = \frac{P \left\{ S = 1, \sum_{i=1}^{20} X_i = 5 \right\}}{P \left\{ \sum_{i=1}^{20} X_i = 5 \right\}} \]

\[ P \left\{ \sum_{i=1}^{20} X_i = 5 \left| S = 1 \right. \right\} \cdot P \left\{ S = 1 \right\} \]

\[ P \left\{ \sum_{i=1}^{20} X_i = 5 \left| S = 1 \right. \right\} \cdot P \left\{ S = 1 \right\} + P \left\{ \sum_{i=1}^{20} X_i = 5 \right\} \cdot P \left\{ S = 0 \right\} \]
EXAMPLE: SOFT PRETZELS

\[ P \left\{ \sum_{i=1}^{20} X_i = 5 \mid S = 1 \right\} \]

0.178 from the binomial table

is the binomial probability that 5 out of 20 people prefer the new pretzel with \( p = 0.3 \)

\[ P \left\{ \sum_{i=1}^{20} X_i = 5 \mid S = 0 \right\} \]

0.032 from the binomial table

is the binomial probability that 5 out of 20 people prefer the new pretzel with \( p = 0.1 \)
EXAMPLE: SOFT PRETZELS

Therefore,

\[
P \left\{ \sum_{i=1}^{20} X_i = 5 \mid \bar{S} = 1 \right\} \cdot P \left\{ \bar{S} = 1 \right\} \\
\frac{\sum_{i=1}^{20} X_i = 5 \mid \bar{S} = 1}{P \left\{ \sum_{i=1}^{20} X_i = 5 \mid \bar{S} = 1 \right\} + P \left\{ \sum_{i=1}^{20} X_i = 5 \mid \bar{S} = 0 \right\} \cdot P \left\{ \bar{S} = 0 \right\}}
\]

\[
= \frac{(0.178)(0.5)}{(0.178)(0.5) + (0.032)(0.5)}
\]

\[
= 0.848
\]
THE POISSON DISTRIBUTION

- The **binomial distribution** is appropriate for the representation of successes in repeated trials.

- The **Poisson distribution** is appropriate for the representation of specific events over time, space, or some other problem-specific dimension, e.g., the number of customers who are served by a butcher in a meat market, or the number of chips judged unacceptable in a production run.
REQUIREMENTS FOR A POISSON DISTRIBUTION

- Events can happen at any of a large number of values within the range of measurement (time, space, etc.) and possibly along a continuum.

- At a specific point $z$, $P\{\text{an event at } z\}$ is very small and therefore events do not happen too frequently.
REQUIREMENTS FOR A POISSON DISTRIBUTION

- Each event is independent of any other event and

\[ P \{ \text{event at any point} \} \]

is constant and independent of all other events

- In fact, the average number of events over a unit of measure is constant
THE POISSON DISTRIBUTED $r.v.$

- $X$ is the $r.v.$ representing the number of events in a unit of measure.

$$P\{X = k\} = \frac{e^{-m} m^k}{k!}$$

$$E\{X\} = m \quad \text{var}\{X\} = m$$

- Interpretation: the Poisson distribution parameter $m$ is the mean or the variance of the distribution.
EXAMPLE: POISSON DISTRIBUTION

Consider an assembly line for manufacturing a particular product

1,024 units are produced

Based on past experience, a flawed unit is produced every 197 units and so, on average, there are \( \frac{1,024}{197} \approx 5.2 \) flawed units in the 1,024 units of the product produced.
EXAMPLE: POISSON DISTRIBUTION

- Note that the Poisson conditions are satisfied
  - the sample has 1,024 units
  - there are only a few flawed units in the 1,024 sample, i.e., the event of the a flawed unit is infrequent
  - the probability of a flawed unit is rather small
  - each flawed unit is independent of every other flawed unit
EXAMPLE: POISSON DISTRIBUTION

- Poisson distribution is appropriate representation with \( m = 5.2 \) and so,

\[
P \{ X = k \} = e^{-5.2} \frac{(5.2)^k}{k!}
\]

- If we want to determine \( P \{ 4 \text{ or more flawed units} \} \), we compute
EXAMPLE: POISSON DISTRIBUTION

\[ P\{X > 4\} = 1 - P\{X \leq 4\} = 1 - 0.406 = 0.594 \]

- lookup Poisson table for \( k = 4, m = 5.2 \)

- The Poisson table states that for \( k = 12, m = 5.2 \)

\[ P\{X \leq 12\} = 0.997 \]

and therefore

\[ P\{X > 12\} = 1 - P\{X \leq 12\} = 0.003 \]
EXAMPLE: SOFT PRETZELS

- The pretzel enterprise is going well: several retail outlets and a street vendor are selling the pretzels.

- A vendor in a new location can sell, on average, 20 pretzels per hour; the vendor in an existing location sells 8 pretzels per hour.
A decision is made to try to set up a second street vendor at a different, new location.

New location is classified along three distinct categories with the given probabilities:

<table>
<thead>
<tr>
<th>category</th>
<th>characterization</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>“good”</td>
<td>20 p/h are sold</td>
<td>0.7</td>
</tr>
<tr>
<td>“bad”</td>
<td>10 p/h are sold</td>
<td>0.2</td>
</tr>
<tr>
<td>“dismal”</td>
<td>6 p/h are sold</td>
<td>0.1</td>
</tr>
</tbody>
</table>
EXAMPLE: SOFT PRETZELS

- After the first week, a long enough period to make a mark, a 30-minute test is run and 7 pretzels are sold during the 30-minute test period

- We analyze the situation and define the r.v. $L = \begin{cases} 
"good" & 10 & p. sold \text{ during test period} \\
"bad" & 5 & p. sold \text{ during test period} \\
"dismal" & 3 & p. sold \text{ during test period} 
\end{cases}$

and assume Poisson distribution applies
EXAMPLE: SOFT PRETZELS

☐ We determine the conditional probabilities of the new location conditioned on the 30-minute test outcomes and evaluate

\[ P\left\{ \tilde{L} = "good" \mid \tilde{X} = 7 \right\}, \ P\left\{ \tilde{L} = "bad" \mid \tilde{X} = 7 \right\} \text{ and} \]

\[ P\left\{ \tilde{L} = "dismal" \mid \tilde{X} = 7 \right\} \]

☐ We compute the values of the Poisson distributed

\[ P\left\{ \tilde{X} = 7 \mid \tilde{L} = "good" \right\} = \frac{e^{-10}(10)^7}{7!} = 0.09 \]
EXAMPLE: SOFT PRETZELS

\[ P\left\{ X = 7 \mid L = " \text{bad} " \right\} = \frac{e^{-5}(5)^7}{7!} = 0.104 \]

\[ P\left\{ X = 7 \mid L = " \text{dismal} " \right\} = \frac{e^{-3}(3)^7}{7!} = 0.022 \]

\[ \square \text{ Then } \quad P\left\{ L = " \text{good} " \mid X = 7 \right\} = \]

\[ P\left\{ X = 7 \mid L = " \text{good} " \right\} \cdot P\left\{ L = " \text{good} " \right\} \]

\[ P\left\{ X = 7 \mid L = " \text{good} " \right\} \cdot P\left\{ L = " \text{good} " \right\} + P\left\{ X = 7 \mid L = " \text{bad} " \right\} \]

\[ P\left\{ L = " \text{bad} " \right\} + P\left\{ X = 7 \mid L = " \text{dismal} " \right\} \cdot P\left\{ L = " \text{dismal} " \right\} \]
EXAMPLE: SOFT PRETZELS

\[ P\{L=\text{"good"} \mid \tilde{X} = 7\} = \frac{(0.09)(0.7)}{(0.09)(0.7)+(0.104)(0.2)+(0.022)(0.1)} \]

\[ = 0.733 \]

Similarly,

\[ P\{L=\text{"bad"} \mid \tilde{X} = 7\} = 0.242 \]

\[ P\{L=\text{"dismal"} \mid \tilde{X} = 7\} = 1 - (0.733 + 0.242) = 0.025 \]
EXponentially Distributed r.v.

Unlike the discrete Poisson or the binomial distributed r.v.s, the exponentially distributed r.v. is continuous.

The density function has the form

\[ f_T(t) = me^{-mt} \]
The exponentially distributed r.v. is related to the Poisson distribution in the following manner.

Consider the Poisson distributed r.v. $X$, where $X$ represents the number of events in a specified quantity of measure, e.g., period of time.

We define $T$ to be the r.v. for the uncertain quantity we measure, e.g., the time between 2 sequential events or the distance between 2 accidents.
Then, $T$ has the exponential distribution with

$$F_T(t) = P\{T \leq t\} = 1 - e^{-mt},$$

$$E\{T\} = \frac{1}{m} \quad \text{and} \quad \text{var}\{T\} = \frac{1}{m^2}$$

The exponentially distributed r.v. is completely specified by the parameter $m$.
EXAMPLE: SOFT PRETZELS

- We know that it takes 3.5 minutes to bake a pretzel and we wish to determine the probability that the next customer will arrive after the pretzel baking is completed, i.e., \( P\{T > 3.5 \text{ minutes}\} \)

- We also are given that the location types are classified as being

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EXAMPLE: SOFT PRETZELS

“good” location $\iff m = 20 \text{ pretzels / hour}$

“bad” location $\iff m = 10 \text{ pretzels / hour}$

“dismal” location $\iff m = 6 \text{ pretzels / hour}$

We compute the probability by conditioning on the location type to obtain
EXAMPLE: SOFT PRETZELS

\[ P \{ T > 3.5 \text{minutes} \} = P \{ T > 3.5 \text{minutes} \mid m = 20 \} \ P \{ m = 20 \} + \]

\[ P \{ T > 3.5 \text{minutes} \mid m = 10 \} \ P \{ m = 10 \} + \]

\[ P \{ T > 3.5 \text{minutes} \mid m = 6 \} \ P \{ m = 6 \} \]

\[ \equiv 0.0583 \text{hour} \]

We evaluate

\[ P \{ T > 3.5m \} = \]
EXAMPLE: SOFT PRETZELS

\[ e^{-0.0583(20)} P\{m = 20\} + e^{-0.0583(10)} P\{m = 10\} + e^{-0.0583(6)} P\{m = 6\} \]

\[
\begin{align*}
P\{m = 20\} &= P\{L = "good" \mid \tilde{X} = 7\} = 0.733 \\
P\{m = 10\} &= P\{L = "bad" \mid \tilde{X} = 7\} = 0.242 \\
P\{m = 6\} &= P\{L = "dismal" \mid \tilde{X} = 7\} = 0.025
\end{align*}
\]
EXAMPLE: SOFT PRETZELS

and so

\[ P\{T > 3.5 \text{ minutes}\} = 0.3809 \]

Therefore,

\[ P\{T \leq 3.5 \text{ minutes}\} = 1 - 0.3809 = 0.6191 \]

and the interpretation is that the majority of the customers arrives before the pretzels are baked.
THE NORMAL DISTRIBUTION

The *normal* or *Gaussian* distribution is, by far, the most important probability distribution since the *Law of Large Numbers* implies that the distribution of many uncertain variables is governed by the *normal* distribution, or more commonly known as the *bell curve*.

We consider a normally distributed r.v. $Y$

\[ Y \sim \mathcal{N}(\mu, \sigma) \]
The density function is

\[ f_Y(y) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\left( \frac{1}{2} \frac{(y-\mu)^2}{\sigma^2} \right)} \]

with \( E\{Y\} = \mu \) and \( \text{var}\{Y\} = \sigma^2 \)
Consider the r.v. $\tilde{Z}$ which has the standard normal distribution

$$Z \sim \mathcal{N}(0,1)$$

1. The relationship between the r.v.s $\tilde{Y}$ and $\tilde{Z}$ is given by the affine expression:

$$Z = \frac{Y - \mu}{\sigma}$$

with

$$P\{\tilde{Y} \leq a\} = P\left\{\tilde{Z} \leq \frac{(a - \mu)}{\sigma}\right\}$$
Note that

\[ E \{ Z \} = 0 \quad \text{and} \quad \text{var} \{ Z \} = 1 \]

In general, any value of the normal distribution is obtained from the standard normal distribution with the affine transformation

\[ Z = \frac{Y - \mu}{\sigma} \]
EXAMPLE: QUALITY CONTROL

❑ We consider a disk drive manufacturing process in which a particular machine produces a part used in the final assembly; the part must rigorously meet the width requirements to be within the interval $[3.995, 4.005]$ mm; else, the company incurs $10.40 in repair costs

❑ The machine is set to produce parts with the width of 4 mm, but in actual conditions, the width is a normally distributed r.v. $W$ with
EXAMPLE: QUALITY CONTROL

\[ W \sim \mathcal{N}(4, \sigma) \]

and

\[ \sigma = f(\text{speed of machine}) = \begin{cases} 
0.0019 & \text{slow speed} \\
0.0026 & \text{high speed} 
\end{cases} \]

The respective costs in $ of the disk drive are

- 20.75 \quad \text{slow speed}
- 20.45 \quad \text{high speed}
EXAMPLE: QUALITY CONTROL

- We may interpret the cost data to imply that more disks can be produced at lower costs at the high speed.
- The problem is to select the machine speed to obtain the more cost effective result.
- A decision tree is useful in the analysis of the situation.
EXAMPLE: QUALITY CONTROL

We compute the probability of each outcome

- **high speed**
  - defective: $(20.45 / \text{disk}) + (10.40 / \text{defect}) = 30.85$
  - not defective: $20.45$

- **low speed**
  - defective: $(20.75 / \text{disk}) + (10.40 / \text{defect}) = 31.15$
  - not defective: $20.75$

We compute the probability of each outcome.
LOW — SPEED PROBABILITY EVALUATION

\[
P\{\text{defective disk is produced}\} = \\
\]

\[
P\{\bar{W} < 3.995 \text{ or } \bar{W} > 4.005\} = \\
\]

\[
1 - P\{3.995 \leq \bar{W} \leq 4.005\} = \\
\]

\[
\bar{Z} = \frac{\bar{W} - 4}{0.0019} \\
\]

\[
1 - P\left\{\frac{3.995 - 4}{0.0019} \leq \bar{Z} \leq \frac{4.005 - 4}{0.0019}\right\} =
\]
LOW — SPEED PROBABILITY EVALUATION

\[ 1 - P\{ -2.63 \leq Z \leq 2.63 \} = \]

\[ 1 - \left[ P\{ Z \leq 2.63 \} - P\{ Z \leq -2.63 \} \right] = 0.0086 \]

0.9957 - 0.0043 = 0.9914
\[ P\{\text{defective disk is produced}\} \]

\[ P\{\bar{W} < 3.995 \text{ or } \bar{W} > 4.005\} \]

\[ 1 - P\{3.995 \leq \bar{W} \leq 4.005\} \]

\[ Z = \frac{\bar{W} - 4}{0.0026} \]

\[ 1 - P\left\{\frac{3.995 - 4}{0.0026} \leq Z \leq \frac{4.005 - 4}{0.0026}\right\} \]
HIGH – SPEED PROBABILITY EVALUATION

\[ 1 - P\{ -1.92 \leq Z \leq 1.92 \} = \]

\[ 1 - \left[ P\{ Z \leq 1.92 \} - P\{ Z \leq -1.92 \} \right] = 0.0548 \]

\[ 0.9726 - 0.0274 = 0.9452 \]
We next evaluate the mean cost per disk

\[ E\{\text{cost}/\text{disk|low speed}\} = (0.9914)(20.75) + (0.0086)(31.15) \]

\[ = 20.84 \]

\[ E\{\text{cost}/\text{disk|high speed}\} = (0.9452)(20.45) + (0.0548)(30.85) \]

\[ = 21.02 \]

We summarize the information in the decision tree.
EXAMPLE: QUALITY CONTROL

- **low speed**
  - defective: 0.0086
  - not defective: 0.9914

- **high speed**
  - defective: 0.0548
  - not defective: 0.9452

M_{low} = 20.84
M_{high} = 21.02
CASE STUDY: OVERBOOKING

- *M Airlines* has a commuter plane capable of flying 16 passengers.
- The plane is used on a route for which *M Airlines* charges $225.
- The airliner’s cost structure is based on:

<table>
<thead>
<tr>
<th>Description</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>the fixed costs for each flight</td>
<td>$900</td>
</tr>
<tr>
<td>the variable costs/passenger</td>
<td>$100</td>
</tr>
<tr>
<td>the “no-show” rate</td>
<td>4%</td>
</tr>
</tbody>
</table>
CASE STUDY — OVERBOOKING

- The refund policy specifies that unused tickets are refunded only if a reservation is cancelled 24 hours ahead of the scheduled departure.

- The overbooking policy pays $100 as an incentive to each bumped passenger and refunds the ticket.

- The decision required is to determine how many reservations should the airliner sell on this plane.
SAMPLE CALCULATION FOR THE CASE 18 RESERVATIONS ARE SOLD

**total revenues:** \( R = 225 \cdot 18 = 4,050 \)

**passenger fixed and variable costs:**

\[ C_1 = 900 + 100 \cdot \min\{\text{number of "shows"}, 16\} \] $ 

**bumping costs:**

\[ C_2 = (225 + 100) \cdot \max\{0, \text{number of "shows"} - 16\} \] $

refunds to customers

**total costs:** \( \tilde{C} = C_1 + C_2 \)
CASE STUDY: OVERBOOKING

We evaluate

\[ P \left\{ \text{no. of "shows"} > 16 \mid \text{reservations sold} = 18 \right\} \]

We assume that each reservation is a r.v. \( P_i \) :

\[ P_i = \begin{cases} 
1 & \text{passenger } i \text{ is a "show" with prob. 0.96} \\
0 & \text{passenger } i \text{ is a "no show" with prob. 0.04} 
\end{cases} \]
Given reservations sold = 18, then we need to evaluate

$$P \left\{ \sum_{i=1}^{18} P_i > 16 \mid 18 \text{ reservations} \right\}$$

We first evaluate

$$P \left\{ \sum_{i=1}^{17} P_i > 16 \mid 17 \text{ res} \right\} = P \left\{ \sum_{i=1}^{17} P_i \geq 17 \mid 17 \text{ res} \right\} = P \left\{ \sum_{i=1}^{17} P_i = 17 \mid 17 \text{ res} \right\}$$

binomial r.v. with $p = 0.96$
CASE STUDY: OVERBOOKING

Then,

\[ P \left\{ \sum_{i=1}^{18} P_i > 16 \mid 18 \text{res} \right\} = P \left\{ \sum_{i=1}^{18} P_i \geq 17 \mid 18 \text{res} \right\} = \]

\[ 18 \left( .4996 \right) \left( .04 \right) + \left( .4996 \right) \left( .96 \right) = 0.8359 \]
CASE STUDY: OVERBOOKING

- Given reservations sold = 19, then we compute and show that

\[ P \left\{ \sum_{i=1}^{19} P_i > 16 \mid 19 \text{ res} \right\} = .9616 \]

- We next consider the profits r.v. \( \pi \), where,

\[ \pi = \tilde{R} - \tilde{C} = \tilde{R} - (C_1 + C_2) \]

and evaluate \( E\{\pi\} \) for different values of reservations sold

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CASE STUDY: OVERBOOKING

For reservations = 16

\[
E\{R\} = (16)(225) = 3,600
\]

\[
E\{C_1\} = 900 + 100 \sum_{n=0}^{16} nP\left\{\sum_{i=1}^{16} P_i = n\right\}
\]

\[
= 900 + 100 E\left\{\sum_{n=1}^{16} P_i\right\}
\]

\[
(16)(.96) = 15.36
\]

\[
= 900 + 1,536
\]
CASE STUDY: OVERBOOKING

\[ E\{C_2\} = (225 + 100) \max \left\{ 0, \sum_{i=1}^{16} P_i - 16 \right\} = 0 \]

and so

\[ E\{\pi|16 \text{ res}\} = E\{\sim\} - E\{C\} \]

\[ = E\{\sim\} - E\{C_1 + C_2\} \]

\[ = 3,600 - 2,436 \]

\[ = 1,164 \]
CASE STUDY: OVERBOOKING

For reservations = 17

\[ E\{R\} = (17)(225) = 3,825 \]

\[ E\{C_1\} = 900 + 100 \sum_{n=0}^{16} nP \left\{ \sum_{i=1}^{17} P_i = n \right\} + 100.16 \cdot P \left\{ \sum_{i=1}^{17} P_i = 17 \right\} \]

\[ = 900 + 782.70 + 799.34 \]

\[ = 2,482.04 \]
also,

\[
E\left\{ C_2 \right\} = 325P \left\{ \sum_{i=1}^{17} P_i = 17 \right\}
\]

\[= 325(0.4996)\]

\[= 162.37\]

and so

\[
E\left\{ \pi \mid 17 \text{ res} \right\} = 3,825 - 2,482.04 - 162.37
\]

\[= 1,180.59 > 1,164\]
CASE STUDY: OVERBOOKING

- For reservations = 18

\[ E \{ R \} = (18)(225) = 4,050 \]

\[ E \{ C_1 \} = 900 + 100 \sum_{n=0}^{16} nP \left\{ \sum_{i=1}^{18} \sim_i = n \right\} + 1,600 \cdot P \left\{ \sum_{i=1}^{18} \sim_i > 16 \right\} \]

= 900 + 253.22 + 1,342.89

= 2,496.11
CASE STUDY: OVERBOOKING

\[ E\left\{ C_2 \right\} = 325 \cdot P\left\{ \sum_{i=1}^{18} P_i = 17 \right\} + 650 \cdot P\left\{ \sum_{i=1}^{18} P_i = 18 \right\} = 428.65 \]

\[ .3597 \]

\[ .4796 \]

and

\[ E\left\{ \pi | 18 \text{ res} \right\} = 4,050 - 2,496.11 - 428.65 \]

\[ = 1,125.24 \]

\[ < 1,180.59 \]
CASE STUDY: OVERBOOKING

- We can show that for reservations $= 19$

\[
E\{\pi \mid 19 \text{ res}\} < 1180.59
\]

- We conclude that the profits are maximized for reservations $= 17$ and so any overbooking over that number results in lower profits.