ECE 307 – Techniques for Engineering Decisions

Lecture 9. Review of Combinatorial Analysis

George Gross
Department of Electrical and Computer Engineering
University of Illinois at Urbana-Champaign
Many problems in probability theory may be solved by simply counting the number of ways a certain event may occur.

We review some basic aspects of combinatorial analysis:

- combinations
- permutations
Suppose that two experiments are to be performed:

- Experiment 1 may result in any one of the \( m \) possible outcomes.
- For each outcome of experiment 1, there exist \( n \) possible outcomes of experiment 2.

Therefore, there are \( mn \) possible outcomes of the two experiments.
The *basic principle* to prove this statement is easily done by using exhaustive enumeration: the set of outcomes for the two experiments is listed as:

\[(1, 1), (1, 2), (1, 3), \cdots (1, n) ;\]
\[(2, 1), (2, 2), (2, 3), \cdots (2, n) ;\]
\[\vdots \]
\[(m, 1), (m, 2), (m, 3), \cdots (m, n)\]

where, \((i, j)\) denotes outcome \(i\) in experiment 1 and outcome \(j\) in experiment 2.
EXAMPLE 1: PAIR FORMATION

We need to form pairs of 1 boy and 1 girl by choosing from a group of 7 boys and 9 girls.

There exist a total of \((7)(9) = 63\) possible pairs since there are 7 ways to pick a boy and 9 ways to pick a girl.
For \( r \) experiments with the first experiment having \( n_1 \) possible outcomes; for every outcome of the first experiment, there are \( n_2 \) possible outcomes for the second experiment, and so on.
There are \( \Pi_{i=1}^{r} n_i = n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_r \) possible outcomes for all the \( r \) experiments, i.e., there are \( \Pi_{i=1}^{r} n_i \) possible branches in the illustration – high dimensionality even for a moderate number \( r \) of experiments.
EXAMPLE 2: SUBCOMMITTEE CHOICES

- The executive committee of an *Engineering Open House* function consists of:
  - 3 first year students
  - 4 sophomores
  - 5 juniors
  - 2 seniors

- We need to form a subcommittee of 4 with each year represented: $3 \cdot 4 \cdot 5 \cdot 2 = 120$ different subcommittees are possible.
EXAMPLE 3: LICENSE PLATE

- We consider possible combinations for a six-place license plate with the first three places consisting of letters and the last three places of numbers.

- Number of combinations with repeats allowed is
  \[(26) (26) (26) (10) (10) (10) = 17,576,000\]

- Combination number if no repetition allowed is
  \[(26) (25) (24) (10) (9) (8) = 11,232,000\]
EXAMPLE 4: $n$ POINTS

Consider $n$ points at which we evaluate the function

$$f(i) \in \{0, 1\}, \; i = 1, 2, \ldots, n$$

Therefore, there are $2^n$ possible function values.
A set of 3 objects \{ A, B, C \} may be arranged in 6 different ways:

- BCA
- ABC
- CBA
- BAC
- ACB
- CAB

Each arrangement is called a permutation.

The total number of permutations is derived from the basic principle:

- there are 3 ways to pick the first element
- there are 2 ways to pick the second element
- there is 1 way to pick the third element
PERMUTATIONS

Therefore, there are \(3 \cdot 2 \cdot 1 = 6\) ways to arrange the 3 elements.

In general, a set of \(n\) objects can be arranged into different permutations

\[n! = n (n-1)(n-2) \ldots 1\]
EXAMPLE 5: BASEBALL

- Number of possible batting orders for a baseball team with nine members is

\[ 9! = 362,880 \]

- Suppose that the team, however, has altogether 12 members; the number of possible batting orders is the product of the number of team formations and the number of permutations is

\[ \frac{12!}{3! \cdot 9!} \cdot 9! = \frac{12!}{3!} = 2(11!) = 79,833,600 \]
EXAMPLE 6: CLASSROOM

- A class with 6 boys and 4 girls is ranked in terms of weight; assume that no two students have the same weight.
- There are
  \[10! = 3,628,800\]
  possible rankings.
- If the boys (girls) are ranked among themselves, the number of different possible rankings is
  \[6!4! = 17,280\]
EXAMPLE 7: BOOKS

- A student has 10 books to put on the shelf:
  - 4 EE, 3 Math, 2 Econ, and 1 Decision

- Student arranges books so that all books in each category are grouped together

- There are $4!3!2!1!$ arrangements so that all EE books are first in line, then the Math books, Econ books, and Decision book
EXAMPLE 8: BOOKS

- But, there are $4!$ possible orderings of the subjects

- Therefore, there are

$$\frac{4!4!3!2!1!}{1!} = 6,912$$

permutations of arranging the 10 books
EXAMPLE 9: PEPPER

We wish to determine the number of different letter arrangements in the word PEPPER.

Consider first the letters $P_1 E_1 P_2 P_3 E_2 R$ where we distinguish the repeated letters among themselves: there are $6!$ permutations of the 6 distinct letters.
EXAMPLE 9: PEPPER

- However, if we consider any single permutation of the 6 letters – for example, $P_1 P_2 E_1 P_3 E_2 R$ – provides the same word $PPEPER$ as 11 other permutations if we fail to distinguish between the same letters.

- Therefore, there are $6!$ permutations for distinct letters but only

$$\frac{6!}{3!2!} = 60$$

permutations with repeated letters that are not distinct.
GENERAL STATEMENT

- Consider a set of \( n \) objects in which
  - \( n_1 \) are alike (category 1)
  - \( n_2 \) are alike (category 2)
  - \( \vdots \)
  - \( n_r \) are alike (category \( r \))

- There are
  \[
  \frac{n!}{n_1! n_2! \ldots n_r!}
  \]
  different permutations
EXAMPLE 9: COLORED BALLS

- We have 3 white, 4 red, and 4 black balls which we arrange in a row; similarly colored balls are indistinguishable from each other.

- There are

\[
\frac{11!}{3!4!4!} = 11,550
\]

possible arrangements.
Given \( n \) objects, we form groups of \( r \) objects and determine the number of different groups we can form.

For example, consider 5 objects denoted as \( A, B, C, D \) and \( E \) and form groups of 3 objects:

- we can pick the first item in exactly 5 ways
- we can pick the second item in exactly 4 ways
- we can pick the third item in exactly 3 ways
and, therefore, we can select

\[ 5 \cdot 4 \cdot 3 = 60 \]

possible groups in which the order of the groups is taken into account.

But, if the order of the objects is not of interest, i.e., we ignore that each group of three objects can be arranged in 6 different permutations, the total number of distinct groups is

\[ \frac{5!}{2!3!} = \frac{60}{6} = 10 \]
The objective is to arrange $n$ elements into groups of $r$ elements. We can select groups of $r$ elements in different ways:

$$\frac{n!}{(n-r)!}$$

different ways

But, each group of $r$ has $r!$ permutations.

The number of different combinations is:

$$\frac{n!}{(n-r)!r!}$$
We define the term

\[
\binom{n}{r} \triangleq \frac{n!}{(n-r)!r!}
\]

as the *binomial coefficient of* \(n\) and \(r\)

A binomial coefficient gives the number of possible combinations of \(n\) elements taken \(r\) at a time
EXAMPLE 10: COMMITTEE SELECTION

We wish to select three persons to represent a class of 40: how many groups of 3 can be formed?

There are

\[
\frac{40!}{37!3!} = \frac{40 \cdot 39 \cdot 38}{3 \cdot 2 \cdot 1} = 20 \cdot 13 \cdot 38 = 9,880
\]

possible group selections
EXAMPLE 11: GROUP FORMATION

Given a group of 5 boys and 7 girls, form sets consisting of 2 boys and 3 girls.

There are possible ways to form such groups:

\[
\binom{5}{2} \binom{7}{3} = \frac{5!}{3! \cdot 2!} \cdot \frac{7!}{4! \cdot 3!} = \frac{5 \cdot 4}{2} \cdot \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} = 350
\]
GENERAL COMBINATORIAL IDENTITY

\[
\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}
\]

- number of ways of selecting groups of \(r\) from \(n\)
- number of ways of selecting groups of \(r-1\) from \(n-1\)
- number of ways of selecting groups of \(r\) from \(n-1\)
MULTINOMIAL COEFFICIENTS

Given a set of \( n \) distinct items, form \( r \) distinct groups of respective sizes \( n_1, n_2, \ldots, n_r \) with

\[
\sum_{i=1}^{r} n_i = n
\]

There are possible choices for the first group
MULTINOMIAL COEFFICIENTS

For each choice of the first group, there are
\[
\binom{n-n_1}{n_2}
\]
possible choices for the second group.

We continue with this reasoning and we conclude that there are
\[
\frac{n!}{n_1!n_2! \ldots n_r!}
\]
possible groups.
MULTINOMIAL COEFFICIENTS

The previous conclusion was gained by realizing that

\[
\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\ldots-n_{r-1}}{n_r} = \frac{n!}{(n-n_1)!n_1!}\frac{(n-n_1)!}{(n-n_1-n_2)!n_2!}\cdots\frac{n-n_1-n_2-\ldots-n_{r-1}}{0!n_r!} = \frac{n!}{n_1!n_2!\cdots n_r!}
\]
MULTINOMIAL COEFFICIENTS

Let

\[ n = n_1 + n_2 + n_3 + \ldots + n_r \]

we define the multinomial coefficient

\[ \binom{n}{n_1, n_2, \ldots, n_r} \triangleq \frac{n!}{n_1!n_2!n_3!\ldots n_r!} \]

A multinomial coefficient represents the number of possible allocations of \( n \) distinct objects into \( r \) distinct groups of respective sizes \( n_1, n_2, \ldots, n_r \).
EXAMPLE 12: POLICE

- A police department of a small town has 10 officers.

- The department policy is to have 5 members on street patrol, 2 members at the station and 3 on reserve.

- The number of possible allocations is

\[
\frac{10!}{5!3!2!} = 2,520
\]
EXAMPLE 13: TEAM FORMATION

We need to form two teams, the \( A \) team and the \( B \) team, with each team having 5 boys from a group of 10 boys.

There are

\[
\frac{10!}{5!5!} = 252
\]

possible teams.
EXAMPLE 13: TEAM FORMATION

- Suppose that these two teams are to play against one another

- In this case, the order of the two teams is irrelevant since there is no team \( A \) and team \( B \) per se but simply a division of 10 boys into 2 groups of 5 each

- The number of ways to form the two teams is

\[
\frac{1}{2!} \left( \frac{10!}{5!5!} \right) = 126
\]
EXAMPLE 14: TEA PARTY

- A woman has 8 friends of whom she will invite 5 to a tea party.

- How many choices does she have if 2 of the friends are feuding and refuse to attend together?

- How many choices does she have if 2 of her friends will only attend together?