Lecture 8a. Dynamic Programming

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Systematic approach to solving sequential decision making problems

Salient problem characteristic: ability to separate the problem into stages

Multi-stage problem solving technique
STAGES AND STATES

- We consider the problem to consist of *multiple separable stages*.

- A *stage* is a “point” in time, space, geographic location or a structural element at which we make a decision; each *stage* is associated with one or more *states*.

- A *state* of the system describes a possible configuration of the system in a given *stage*. 
STAGES AND STATES

- **Stage n**
  - **Decision variable** $(d_n)$
  - **State** $(s_n)$
  - **Estimated state** $(\tilde{s}_n)$

- **States**
  - *Input state*
  - *Output state*
A decision $d_n$ in the stage $n$ transforms the state $s_n$ in the stage $n$ into the state $s_{n+1}$ in the stage $n + 1$.

The state $s_n$ and the decision $d_n$ have an impact on the objective function; the effect is measured in terms of the return function denoted by $r_n(s_n, d_n)$.

The optimal decision at stage $n$ is the decision $d^*_n$ that optimizes the return function for the state $s_n$.
RETURN FUNCTION

stage \( n \)

- **decision variable** (decision)
  - \( d_n \)

- **state** (output)
  - \( \tilde{s}_n \)

- **state** (input)
  - \( s_n \)

- **return function**
  - \( r_n(s_n, d_n) \)
ROAD TRIP EXAMPLE

- A poor student is traveling from NY to LA
- To minimize costs, the student plans to sleep at friends’ houses each night in cities along the trip
- Based on past experience he can reach
  - Columbus, Nashville or Louisville after 1 day
  - Kansas City, Omaha or Dallas after 2 days
  - San Antonio or Denver after 3 days
  - LA after 4 days
ROAD TRIP EXAMPLE

- **NY (1)**
  - Day 1: 900 to Columbus
  - Day 2: 770 to Nashville

- **Columbus (2)**
  - Day 3: 680 to K. City

- **K. City (5)**
  - Day 4: 610 to Denver

- **Denver (8)**
  - Day 3: 540 to S. Antonio
  - Day 4: 1030 to LA

- **Nashville (3)**
  - Day 2: 580 to Omaha

- **Omaha (6)**
  - Day 3: 790 to S. Antonio

- **Louisville (4)**
  - Day 2: 830 to Dallas

- **S. Antonio (9)**
  - Day 4: 270 to Dallas

- **Dallas (7)**
  - Day 4: 510 to Omaha
  - Day 3: 1050 to Omaha

- **LA (10)**
  - Day 4: 1390 to LA

**Day 1:**
- NY to Columbus: 900 miles
- NY to Nashville: 770 miles

**Day 2:**
- Columbus to Nashville: 680 miles
- Nashville to Omaha: 580 miles

**Day 3:**
- K. City to S. Antonio: 540 miles
- Omaha to S. Antonio: 790 miles

**Day 4:**
- Denver to LA: 1030 miles
- LA to S. Antonio: 1390 miles

**Total Distance:**
- 3,790 miles
The student wishes to minimize the number of miles driven and so he wishes to determine the shortest path from NY to LA.

To solve the problem, he works backwards.

We adopt the following notation:

\[ c_{i,j} = \text{distance between states } i \text{ and } j \]

\[ f_k(i) = \text{distance of the shortest path to LA from state } i \text{ in the stage } k \]
ROAD TRIP EXAMPLE CALCULATIONS

day 4: \[ f_4(8) = c_{8,10} = 1,030 \quad f_4(9) = c_{9,10} = 1,390 \]

day 3: \[ f_3(5) = \min \left\{ \frac{(610 + 1,030)}{1,640}, \frac{(790 + 1,390)}{2,180} \right\} = 1,640 \]

\[ f_3(6) = \min \left\{ \frac{(540 + 1,030)}{1,570}, \frac{(940 + 1,390)}{2,330} \right\} = 1,570 \]

\[ f_3(7) = \min \left\{ \frac{(790 + 1,030)}{1,820}, \frac{(270 + 1,390)}{1,660} \right\} = 1,660 \]
ROAD TRIP EXAMPLE CALCULATIONS

day 2:

\[ f_2(2) = \min \left\{ \frac{680 + 1,640}{2,320}, \frac{790 + 1,570}{2,360}, \frac{1,050 + 1,660}{2,710} \right\} = 2,320 \]

\[ f_2(3) = \min \left\{ \frac{580 + 1,640}{2,220}, \frac{760 + 1,570}{2,330}, \frac{660 + 1,660}{2,320} \right\} = 2,220 \]

\[ f_2(4) = \min \left\{ \frac{510 + 1,640}{2,150}, \frac{700 + 1,570}{2,270}, \frac{830 + 1,660}{2,490} \right\} = 2,150 \]
ROAD TRIP EXAMPLE

day 1:

\[ f_1 (1) = \min \left\{ (550 + 2,320), (900 + 2,220), (770 + 2,150) \right\} = 2,870 \]

- The shortest path is 2,870 miles and corresponds to the trajectory \( \{ (1, 2), (2, 5), (5, 8), (8, 10) \} \), i.e., from NY, the student reaches Columbus on the first day, Kansas City on the second day, Denver the third day and then LA.

- Every other trajectory to LA leads to higher costs and so is, by definition, suboptimal.
There are 30 matches on a table and 2 players

Each player can pick up 1, 2, or 3 matches and continue until the last match is picked up

The loser is the person who picks up the last match

How can the player $P_1$, who goes first, ensure to be the winner?
WORKING BACKWARDS: PICK UP MATCHES GAME

- We solve this problem by reasoning in a backwards fashion so as to ensure that when a single match remains, \( P_2 \) has the turn.

- Consider the situation where 5 matches remain and it is \( P_2 \)'s turn; for \( P_1 \) to win, we consider all possible situations:
WORKING BACKWARDS: PICK UP MATCHES GAME

\[ P_2 \text{'s move is to pick} \]

\[ \begin{align*}
1 \Rightarrow 4 \text{ left} & \Rightarrow P_1 \text{ removes 3} \\
2 \Rightarrow 3 \text{ left} & \Rightarrow P_1 \text{ removes 2} \\
3 \Rightarrow 2 \text{ left} & \Rightarrow P_1 \text{ removes 1}
\end{align*} \]

- We can reason similarly for the cases of 9, 13, 17, 21, 25, and 29 matches.

- Therefore, \( P_1 \) wins if \( P_1 \) picks \( 30 - 29 = 1 \) match in the first move.

- In this manner, we can assure a win for any number of matches in the game.
OIL TRANSPORT TECHNOLOGY

intermediate region

oil storage

substations

final destinations
We consider the development of a transport network from the north slope of Alaska to one of 6 possible shipping points in the US.

The network must meet the problem feasibility requirements:

- 7 pumping stations from a north slope ground storage plant to a shipping port.
use of only those paths that are physically and environmentally feasible

Objective: determine a feasible pumping configuration that minimizes the total
costs

\[ \text{total costs} = \sum \text{construction costs of branches of network of the feasible pumping configuration} \]
Possible approaches to solving such a problem include:

- **enumeration**: exhaustive evaluation of all possible paths, which is too costly since there are more than 100 possible paths for this small size problem.

- **myopic decision rule**: at each node, pick as the next node the one reachable by the cheapest path (in case of ties the pick is arbitrary); we show a possible path.
but, such a path is not unique and cannot be guaranteed to be \textit{optimal}.

\begin{itemize}
  \item \textit{serial dynamic programming (DP)}: we need to construct the problem solution by defining the \textit{stages, states and decisions}.
\end{itemize}
We define an intermediate *stage* to represent each pumping region and so each such *stage* corresponds to the set of vertical nodes in regions I, II, ..., VII.

We also define a stage of final destinations and the initial stage for oil storage.

We use *backwards recursion*: we start from every final destination and work *backwards* to the oil storage *stage*. 
We define a state $s_k$ to denote a final destination, a specific pumping station in the intermediate regions or the oil storage tank with all the oil.

A decision refers to the selection of the branch from each state $s_k$, so there are at most three choices for a decision $d_k$:

$L \leftrightarrow \text{left} \quad F \leftrightarrow \text{forward} \quad R \leftrightarrow \text{right}$
DP SOLUTION

- The return function $r_k(s_k, d_k)$ is defined as the costs associated with the decision $d_k$ for the state $s_k$.

- The transition function is the total costs in proceeding from a state $s_{k+1}$ in stage $k+1$ to another state $s_k$ in stage $k$, $k = 0, 1, \ldots, 7$.

- We solve the problem by iteratively moving backwards, starting from each final state to the states in stage 1 and so on, until we reach the oil storage.
DP SOLUTION: \textit{STAGE 1} \leftrightarrow \textit{REGION VII} TO A FINAL DESTINATION

Optimal decision

<table>
<thead>
<tr>
<th>$s_1$</th>
<th>$d_1$</th>
<th>$d^*$</th>
<th>$f_1^*(s_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$r$</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>$l$</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$f$</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>$r$</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$l$</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>$f$</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$r$</td>
<td>4</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$l$</td>
<td>6</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$f$</td>
<td>6</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>
**DP SOLUTION:**

**STAGE 2 ↔ REGION VI TO STAGE 1**

<table>
<thead>
<tr>
<th>$s_2$</th>
<th>$d_2$</th>
<th>$d_2^*$</th>
<th>$f_2^*(s_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>10</td>
<td>12</td>
<td>$R$</td>
</tr>
<tr>
<td>B</td>
<td>9</td>
<td>12</td>
<td>$F$</td>
</tr>
<tr>
<td>C</td>
<td>5</td>
<td>6</td>
<td>$R$</td>
</tr>
<tr>
<td>D</td>
<td>8</td>
<td>7</td>
<td>$F$</td>
</tr>
<tr>
<td>E</td>
<td>7</td>
<td>6</td>
<td>$L$</td>
</tr>
</tbody>
</table>

*Optimal decision:

From the state $s_2$ to a final destination, the cumulative costs in proceeding.*
STAGE 2 CALCULATION

costs of proceeding from the state $s_2$ to a state $s_1$ in stage 1

$$f^*_2(s_2) = \min_{d_2} \left( r_2(s_2, d_2) + f^*_1(s_1) \right)$$

a function of only $s_1$

for a given $d_2$, the state $s_1$ is set
**DP SOLUTION:**

**STAGE 3 ↔ REGION V TO STAGE 2**

\[
f^*_3(s_3) = \min_{d_3} \left\{ r_3(s_3, d_3) + f^*_2(s_2) \right\}
\]

<table>
<thead>
<tr>
<th></th>
<th>(d_3)</th>
<th></th>
<th>(d^*_3)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(R)</td>
<td>(L)</td>
<td>(F)</td>
<td></td>
</tr>
<tr>
<td>(A)</td>
<td>14</td>
<td>16</td>
<td>(R)</td>
<td>14</td>
</tr>
<tr>
<td>(B)</td>
<td>14</td>
<td>17</td>
<td>15</td>
<td>14</td>
</tr>
<tr>
<td>(C)</td>
<td>10</td>
<td>5</td>
<td>13</td>
<td>10</td>
</tr>
<tr>
<td>(D)</td>
<td>9</td>
<td>12</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>(E)</td>
<td>12</td>
<td>15</td>
<td>(L)</td>
<td>12</td>
</tr>
</tbody>
</table>

Cumulative costs in proceeding from the state \(s_3\) to a final destination.
### DP SOLUTION:

**STAGE 4 ↔ REGION IV TO STAGE 3**

\[ f^*_4(s_4) = \min_{d_4} \left\{ r_4(s_4, d_4) + f^*_3(s_3) \right\} \]

<table>
<thead>
<tr>
<th></th>
<th>( d_4 )</th>
<th>( d^*_4 )</th>
<th>( f^*_4(s_4) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>R</td>
<td>L</td>
<td>F</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>17</td>
<td>18</td>
<td>23</td>
</tr>
<tr>
<td><strong>C</strong></td>
<td>15</td>
<td>22</td>
<td>16</td>
</tr>
<tr>
<td><strong>D</strong></td>
<td>18</td>
<td>17</td>
<td>16</td>
</tr>
<tr>
<td><strong>E</strong></td>
<td>16</td>
<td>21</td>
<td></td>
</tr>
</tbody>
</table>

*Cumulative costs in proceeding from the state \( s_4 \) to a final destination.*

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\[ f^*_5(s_5) = \min_{d_5} \left\{ r_5(s_5, d_5) + f^*_4(s_4) \right\} \]

<table>
<thead>
<tr>
<th>( s_5 )</th>
<th>( d_5 )</th>
<th>( d^*_5 )</th>
<th>( f^*_5(s_5) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>19</td>
<td>R</td>
<td>19</td>
</tr>
<tr>
<td>B</td>
<td>18</td>
<td>18</td>
<td>R, F</td>
</tr>
<tr>
<td>C</td>
<td>24</td>
<td>23</td>
<td>17</td>
</tr>
<tr>
<td>D</td>
<td>20</td>
<td>19</td>
<td>25</td>
</tr>
<tr>
<td>E</td>
<td>21</td>
<td>17</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>20</td>
<td>L</td>
<td>20</td>
</tr>
</tbody>
</table>
**DP SOLUTION:**

**STAGE 6 ↔ REGION II TO STAGE 6**

\[
f_6^*(s_6) = \min_{d_6} \left\{ r_6(s_6, d_6) + f_5^*(s_5) \right\}
\]

<table>
<thead>
<tr>
<th></th>
<th>(d_6)</th>
<th>(d^*_6)</th>
<th>(f_6^*(s_6))</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>s_6</strong></td>
<td>R</td>
<td>L</td>
<td>F</td>
</tr>
<tr>
<td><strong>A</strong></td>
<td>25</td>
<td>24</td>
<td>F</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>21</td>
<td>25</td>
<td>24</td>
</tr>
<tr>
<td><strong>C</strong></td>
<td>28</td>
<td>21</td>
<td>23</td>
</tr>
<tr>
<td><strong>D</strong></td>
<td>27</td>
<td>26</td>
<td>29</td>
</tr>
<tr>
<td><strong>E</strong></td>
<td>26</td>
<td>23</td>
<td>22</td>
</tr>
<tr>
<td><strong>F</strong></td>
<td>18</td>
<td>23</td>
<td>L</td>
</tr>
</tbody>
</table>
**DP SOLUTION:**

**STAGE 6 ↔ REGION II TO STAGE 6**

\[
 f^*_{7}(s_7) = \min_{d_7} \left\{ r_{7}(s_7, d_7) + f^*_{6}(s_6) \right\}
\]

<table>
<thead>
<tr>
<th>(s_7)</th>
<th>(d_7)</th>
<th>(d^*_7)</th>
<th>(f^*_{7}(s_7))</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>R 27</td>
<td>L 32</td>
<td>R 27</td>
</tr>
<tr>
<td>B</td>
<td>R 26</td>
<td>L 33</td>
<td>R,F 26</td>
</tr>
<tr>
<td>C</td>
<td>R 34</td>
<td>L 25</td>
<td>L 25</td>
</tr>
<tr>
<td>D</td>
<td>R 25</td>
<td>L 27</td>
<td>R 25</td>
</tr>
<tr>
<td>E</td>
<td>R 27</td>
<td>L 35</td>
<td>R 27</td>
</tr>
</tbody>
</table>

Cumulative costs in proceeding from the state \(s_7\) to a final destination.
The optimal trajectory

The last stage consists of only 1 state – the oil storage.

<table>
<thead>
<tr>
<th>$s_8$</th>
<th>$d_8$</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>$d_8^*$</th>
<th>$f_8^*(s_8)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>33</td>
<td>30</td>
<td>32</td>
<td>33</td>
<td>30</td>
<td>B,E</td>
<td>30</td>
</tr>
</tbody>
</table>

\[
f_8^*(s_8) = \min \{ 27 + 6, 26 + 4, 25 + 7, 25 + 8, 27 + 3 \}
\]

\[
= 30
\]

To find the optimal trajectory, we retrace in a forward direction to go through the stages 7, 6, \ldots, 1 to get

The least-cost trajectory that terminates in shipping point $D$.
Besides this *optimal* solution, other trajectories are possible since the path need not be unique but no path yields a shorter total distance.
OIL TRANSPORT PROBLEM
SOLUTION

- We obtain the diagram below by retracing the steps of proceeding to an endpoint at each stage.

- The solution provides all the optimal trajectories.

- It is based on logically breaking up the problem into stages with calculations in each stage being a function of the number of states in that stage.

- It provides also all the suboptimal paths.
OIL TRANSPORT PROBLEM
SOLUTION

oil storage

substations

final destinations
For example, we may calculate the least cost optimal path to any sub-optimal shipping point different than $D$.

From the solution, we can also determine the sub-optimal path if the construction of a feasible path is not undertaken.
Consider the case where we got to stage VI but the branch VI – D to VII – D cannot be built due to some newly-enacted environmental constraint.

We then determine the least-cost path from VI – D to find the final destination D whose value is 9 instead of 6.

And so the sub optimal cost solution costs are 33.
FACILITIES SELECTION PROBLEM

- A company is expanding to meet a wider market and considers:
  - 3 location alternatives
  - 4 different building types (sizes) at each site

- Revenues and costs vary with each location and building type
FACILITIES SELECTION PROBLEM

- Revenues $R$ increase monotonically with building size; please note that we are referring to net revenues or profits.

- Costs $C$ increase monotonically with building size.

- The data for building sizes and the associated revenues and costs are given in the table.
## FACILITIES SELECTION PROBLEM

<table>
<thead>
<tr>
<th>site</th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$B_3$</th>
<th>$B_4$</th>
<th>none</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$R_1$</td>
<td>$c_1$</td>
<td>$R_2$</td>
<td>$c_2$</td>
<td>$R_3$</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>1</td>
<td>0.65</td>
<td>2</td>
<td>0.8</td>
</tr>
<tr>
<td>II</td>
<td>0.62</td>
<td>2</td>
<td>0.78</td>
<td>5</td>
<td>0.96</td>
</tr>
<tr>
<td>III</td>
<td>0.71</td>
<td>4</td>
<td>1.2</td>
<td>7</td>
<td>1.6</td>
</tr>
</tbody>
</table>
The company can afford to invest at most $21 million in the total expansion project.

The goal is to determine the optimal expansion policy, i.e., the buildings to be built at each site.
We use the DP approach to solve this problem, but first, we must define the DP structure elements.

For the facilities siting problem, we realize that without the choice of a site, the building type is irrelevant and so the elements that control the entire decision process are the building sites.
**DP** SOLUTION APPROACH

<table>
<thead>
<tr>
<th>stage</th>
<th>⇔</th>
<th>site</th>
</tr>
</thead>
<tbody>
<tr>
<td>amount of funds available</td>
<td></td>
<td>for construction</td>
</tr>
<tr>
<td>decision</td>
<td>⇔</td>
<td>building type</td>
</tr>
<tr>
<td>return function</td>
<td>⇔</td>
<td>revenues</td>
</tr>
<tr>
<td>transition function</td>
<td>⇔</td>
<td>impact of a decision on the available funds</td>
</tr>
</tbody>
</table>
We use backwards DP to solve the problem and start with site 1 ↔ stage 1, a purely arbitrary choice, where this stage 1 represents the last decision in the 3 – stage sequence and so is made after the decision for the other two sites have been taken.
The amount of funds available is unknown since the decision at sites II and III are already made, and so

\[ 0 \leq s_1 \leq 21 \]

There are no additional decisions to be made in stage 0 and we define

\[ s_{o} = 0 \quad \text{and} \quad f_{o}^{*}(s_{o}) = 0 \]
DP SOLUTION APPROACH

- We start with stage 1 and move backwards to stages 2 and 3.

- As we move backwards from stage \((n - 1)\) to stage \(n\), as a result of the decision \(d_n\), the funds available for construction in stage \((n - 1)\) are

\[
S_{n-1} = S_n - c_n \quad \text{costs of decision } d_n
\]
The recursion relation is given by

$$f^*_n(s_n) = \max_{d_n} \left\{ f_n(s_n, d_n) + f^*_{n-1}(s_{n-1}) \right\}, \quad n = 1, 2, 3$$

with

$$s_{n-1} = s_n - c_n$$

and

$$f_n(s_n, d_n) = r_n(s_n, d_n) = R_n$$

revenues for decision $d_n$
**DP SOLUTION: STAGE 1 ↔ SITE I**

\[
f^*_1(s_1) = \max_{0 \leq d_1 \leq 4} \left\{ r_1(s_1, d_1) \right\}
\]

<table>
<thead>
<tr>
<th>(s_1)</th>
<th>(d_1)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>(d^*_1)</th>
<th>(f^*_1(s_1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>21 ≥ (s_1) ≥ 5</td>
<td>0</td>
<td>0.5</td>
<td>0.65</td>
<td>0.8</td>
<td>1.4</td>
<td>4.0</td>
<td>4</td>
<td>1.4</td>
</tr>
<tr>
<td>4 ≥ (s_1) ≥ 3</td>
<td>0</td>
<td>0.5</td>
<td>0.65</td>
<td>0.8</td>
<td>1.4</td>
<td></td>
<td></td>
<td>0.8</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0.5</td>
<td>0.65</td>
<td></td>
<td>2.0</td>
<td>2.0</td>
<td></td>
<td>0.65</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0.5</td>
<td></td>
<td></td>
<td>1.0</td>
<td>1.0</td>
<td></td>
<td>0.5</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td></td>
<td>0.0</td>
</tr>
</tbody>
</table>
DP SOLUTION: STAGE 2 ↔ SITE II

- The amount of funds $s_2$ available is unknown since the decision at site III is already made.
- The value of $d_2$ is a function of $s_2$ and we construct a decision table using

$$f^*_2(s_2) = \max \left\{ r_2(s_2, d_2) + f^*_1(s_1) \right\}_{R_2}$$

where

$$0 \leq d_2 \leq 4$$

$$s_1 = s_2 - c_2$$
**DP SOLUTION: STAGE 2 ↔ SITE II**

<table>
<thead>
<tr>
<th>$s_2$</th>
<th>$d_2$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>$d_2^*$</th>
<th>$f_2^*(s_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>21 ≥ $s_2$ ≥ 13</td>
<td>12</td>
<td>1.40</td>
<td>2.02</td>
<td>2.18</td>
<td>2.36</td>
<td>3.20</td>
<td>4</td>
<td>3.20</td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>1.40</td>
<td>2.02</td>
<td>2.18</td>
<td>2.36</td>
<td>2.60</td>
<td>4</td>
<td>2.60</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>1.40</td>
<td>2.02</td>
<td>2.18</td>
<td>1.76</td>
<td>2.45</td>
<td>4</td>
<td>2.45</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>1.40</td>
<td>2.02</td>
<td>1.58</td>
<td>1.61</td>
<td>2.30</td>
<td>4</td>
<td>2.30</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>1.40</td>
<td>2.02</td>
<td>1.58</td>
<td>1.61</td>
<td>1.80</td>
<td>1</td>
<td>2.02</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>1.40</td>
<td>2.02</td>
<td>1.43</td>
<td>1.46</td>
<td></td>
<td>1</td>
<td>2.02</td>
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<tr>
<td></td>
<td>6</td>
<td>1.40</td>
<td>1.42</td>
<td>1.28</td>
<td>0.96</td>
<td></td>
<td>1</td>
<td>1.42</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>1.40</td>
<td>1.42</td>
<td>0.78</td>
<td></td>
<td></td>
<td>1</td>
<td>1.42</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.80</td>
<td>1.27</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td>1.27</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.80</td>
<td>1.12</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td>1.12</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.65</td>
<td>0.62</td>
<td></td>
<td></td>
<td></td>
<td>0</td>
<td>0.65</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.50</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0</td>
<td>0.50</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0</td>
<td>0.00</td>
</tr>
</tbody>
</table>
SAMPLE CALCULATIONS

Consider the case \( s_2 = 10 \) and \( d_2 = 0 \); then,

\[
c_2 = 0 \quad \text{and} \quad R_2 = 0
\]

and so,

\[
s_1 = 10 \quad \text{and} \quad d_1^* = 4
\]

Therefore,

\[
f_1^*(s_1) = 1.4
\]

and consequently,

\[
f_2(s_2) = 1.4
\]
Consider next the case $s_2 = 10$ and $d_2 = 4$; then,

$$c_2 = 8 \text{ and } R_2 = 1.8$$

and so,

$$s_1 = 2$$

so that

$$f^* (s_1) = 0.65$$

Consequently,

$$f_2 (s_2) = 2.45$$

which we can show is the optimal value, so that

$$f^*_2 (s_2) = 2.45$$
At stage 3, the first decision is actually taken and so exactly 21 million is available and \( s_3 = 21 \).

We compute the elements in the table using

\[
f^*_{3} (s_3) = \max_{d_3} \left\{ r_3(s_3, d_3) + f^*_{2}(s_2) \right\}
\]

where

\[
s_2 = s_3 - c_3
\]
**OPTIMAL SOLUTION**

<table>
<thead>
<tr>
<th>$s_3$</th>
<th>$d_3$</th>
<th>$d^*_3$</th>
<th>$f^*_3(s_3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>3.20</td>
<td>3.91</td>
<td>4.40</td>
</tr>
<tr>
<td></td>
<td>4.40</td>
<td>4.20</td>
<td>4.45</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>4.45</td>
</tr>
</tbody>
</table>

- **Optimal** profits are 4.45 million and the **optimal** path is obtained by retracing the steps from stage 3 to stage 1 in the forward direction:
OPTIMAL SOLUTION

\[ d_3^* = 4 \quad \leftrightarrow \quad \text{construct } B_4 \text{ at site III} \]

\[ s_2 = s_3 - c_3 = 21 - 11 = 10 \]

\[ d_2^* = 4 \quad \leftrightarrow \quad \text{construct } B_4 \text{ at site II} \]

\[ s_1 = s_2 - c_2 = 10 - 8 = 2 \]

\[ d_1^* = 2 \quad \leftrightarrow \quad \text{construct } B_2 \text{ at site I} \]

\[ c_1 = 5 \quad \text{and} \quad c_1 + c_2 + c_3 = 21 \]
We next consider the case where the maximum investment available is 15 million.

By inspection, the results in stages 1 and 2 remain unchanged; however, we must recompute stage 3 results with the 15 million limit.

<table>
<thead>
<tr>
<th>$s_3$</th>
<th>$d_3$</th>
<th>$d_3^*$</th>
<th>$f^*_3(s_3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>3.2</td>
<td>3.31</td>
<td>3.06</td>
</tr>
<tr>
<td></td>
<td>3.22</td>
<td>3.06</td>
<td>3.27</td>
</tr>
<tr>
<td></td>
<td>3.27</td>
<td>1</td>
<td>3.31</td>
</tr>
</tbody>
</table>
The optimal solution obtains maximum profits of 3.31 million and the decision is as follows:

\[ d_3^* = 1 \iff \text{construct } B_1 \text{ at site III} \]

\[ s_2 = s_3 - c_3 = 15 - 4 = 11 \]

\[ d_2^* = 4 \iff \text{construct } B_4 \text{ at site II} \]

\[ s_1 = s_2 - c_2 = 11 - 8 = 3 \]

\[ d_1^* = 3 \iff \text{construct } B_3 \text{ at site I} \]

\[ c_1 = 3 \text{ and } c_1 + c_2 + c_3 = 15 \]