ECE 307 – Techniques for Engineering Decisions

Lecture 5. Networks and Flows

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A network is a system of lines or channels or branches that connect different points.

Examples abound in nearly all aspects of life:

- electrical systems;
- communication networks;
- airline webs;
- local area networks; and
- distribution systems.
The network structure is also common to many other systems that at first glance are not necessarily viewed as networks:

- distribution of products through a system consisting of manufacturing plants, warehouses and retail outlets
- matching problems such as work to people, tasks to machines and computer dating
NETWORKS AND FLOWS

- river systems with pondage for electricity generation
- mail delivery networks
- freight delivery networks
- project management of multiple tasks in a large undertaking such as a major construction project or a space flight

- We consider a broad range of network and network flow problems
The basic idea of the transportation problem is illustrated with the problem of the distribution of a specified \textit{homogeneous} product from several warehouses to a number of localities \textit{at least cost}.

We consider a system with $m$ warehouses, $n$ markets and links between them with the specified costs of transportation.
THE TRANSPORTATION PROBLEM

Supply

\[ \begin{align*}
W_1 & \quad a_1 \quad M_1 \\
W_2 & \quad a_2 \quad M_2 \\
W_i & \quad a_i \\
W_m & \quad a_m \quad M_n
\end{align*} \]

Transportation links with costs

\[ c_{ij} \]

Whenver warehouse \( i \) cannot ship to market \( j \)

\[ c_{ij} = \infty \]

Demand

\[ \begin{align*}
b_1 \\
b_2 \\
b_j \\
b_n
\end{align*} \]
THE TRANSPORTATION PROBLEM

- all the supply comes from the \( m \) warehouses; we associate the index \( i = 1, 2, \ldots, m \) with a warehouse

- all the demand is at the \( n \) markets; we associate the index \( j = 1, 2, \ldots, n \) with a market

- shipping costs \( c_{ij} \) for each unit from the warehouse \( i \) to the market \( j \)
The transportation problem is to determine the optimal shipping schedule that minimizes shipping costs from the set of \( m \) warehouses to the set of \( n \) markets by determining the quantities shipped from each warehouse \( i \) to each market \( j \),

\[ i = 1, 2, \ldots, m \quad \text{and} \quad j = 1, 2, \ldots, n \]
The decision variables are defined to be

\[ x_{i,j} = \text{quantity shipped from warehouse } i \text{ to market } j, \]

\[ i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n \]

The objective function is

\[
\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}
\]
The constraints are:

\[
\sum_{j=1}^{n} x_{ij} \leq a_i \quad i = 1, 2, \ldots, m
\]

\[
\sum_{i=1}^{m} x_{ij} \geq b_j \quad j = 1, 2, \ldots, n
\]

\[
x_{ij} \geq 0 \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n
\]
Note that feasibility requires that
\[ \sum_{i=1}^{m} a_i \geq \sum_{j=1}^{n} b_j \]

When
\[ \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \]
all available supply at the \( m \) warehouses is shipped to meet all the demands of the \( n \) markets; this is known as the standard transportation problem.
STANDARD TRANSPORTATION PROBLEM (STP)

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

s.t.

$$\sum_{j=1}^{n} x_{ij} = a_i \quad \left\{ \begin{array}{l} i = 1, \ldots, m \\ j = 1, \ldots, n \end{array} \right.$$ 

$$\sum_{i=1}^{m} x_{ij} = b_j \quad \geq 0$$
The standard transportation problem has

- $mn$ variables $x_{ij}$
- $m + n$ equality constraints

However, since

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

there are at most $(m + n - 1)$ independent constraints and consequently at most $(m + n - 1)$ independent variables $x_{ij}$ (basic variables)
## TRANSPORTATION PROBLEM

### EXAMPLE

<table>
<thead>
<tr>
<th>Market $j$</th>
<th>$M_1$</th>
<th>$M_2$</th>
<th>$M_3$</th>
<th>$M_4$</th>
<th>Supplies</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_1$</td>
<td>$x_{11}$</td>
<td>$x_{12}$</td>
<td>$x_{13}$</td>
<td>$x_{14}$</td>
<td>$a_1$</td>
</tr>
<tr>
<td></td>
<td>$c_{11}$</td>
<td>$c_{12}$</td>
<td>$c_{13}$</td>
<td>$c_{14}$</td>
<td></td>
</tr>
<tr>
<td>$W_2$</td>
<td>$x_{21}$</td>
<td>$x_{22}$</td>
<td>$x_{23}$</td>
<td>$x_{24}$</td>
<td>$a_2$</td>
</tr>
<tr>
<td></td>
<td>$c_{21}$</td>
<td>$c_{22}$</td>
<td>$c_{23}$</td>
<td>$c_{24}$</td>
<td></td>
</tr>
<tr>
<td>$W_3$</td>
<td>$x_{31}$</td>
<td>$x_{32}$</td>
<td>$x_{33}$</td>
<td>$x_{34}$</td>
<td>$a_3$</td>
</tr>
<tr>
<td></td>
<td>$c_{31}$</td>
<td>$c_{32}$</td>
<td>$c_{33}$</td>
<td>$c_{34}$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Demands</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_3$</th>
<th>$b_4$</th>
</tr>
</thead>
</table>

$$\sum_i a_i = \sum_j b_j$$
### TRANSPORTATION PROBLEM
#### NUMERICAL EXAMPLE

<table>
<thead>
<tr>
<th></th>
<th>$M_1$</th>
<th>$M_2$</th>
<th>$M_3$</th>
<th>$M_4$</th>
<th>$a_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_1$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>$W_2$</td>
<td>10</td>
<td>8</td>
<td>5</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>$W_3$</td>
<td>7</td>
<td>6</td>
<td>6</td>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td>$b_j$</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>
THE LEAST – COST RULE
PROCEDURE

This procedure generates an initial basic feasible solution which has at most \((m + n - 1)\) positive-valued basic variables.

The principal idea of the scheme is to select, at each step, the variable \(x_{ij}\) with the lowest shipping costs \(c_{ij}\) as the next basic variable to enter the basis.
APPLICATION OF THE LEAST – COST RULE

- $c_{14}$ is the lowest $c_{ij}$ and we select $x_{14}$ as a basic variable

- We choose $x_{14}$ as large as possible without violating any constraints:
  
  $$\min \{ a_1, b_4 \} = \min \{ 3, 4 \} = 3$$

- We set $x_{14} = 3$ and
  
  $$x_{11} = x_{12} = x_{13} = 0$$

- We delete row 1 from any further consideration since all the supplies from $W_1$ are exhausted
### APPLICATION OF THE LEAST – COST RULE

<table>
<thead>
<tr>
<th>w/h i</th>
<th>( M_1 )</th>
<th>( M_2 )</th>
<th>( M_3 )</th>
<th>( M_4 )</th>
<th>( a_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W_1 )</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>( W_2 )</td>
<td>10</td>
<td>8</td>
<td>5</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>( W_3 )</td>
<td>7</td>
<td>6</td>
<td>6</td>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td>( b_j )</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>
APPLICATION OF THE LEAST – COST RULE

- The remaining demand at $M_4$ is

$$4 - 3 = 1$$

which is the value for the modified demand at $M_4$

- We again apply the *criterion selection* for the reduced tableau: since $c_{24}$ is the lowest-valued $c_{ij}$, we select $x_{24}$ as the next *basic variable*.
APPLICATION OF THE LEAST – COST RULE

- We wish to set $x_{24}$ as large as possible without violating any constraints:

$$\min \{ a_2, b_4 \} = \min \{ 7, 1 \} = 1$$

and we set $x_{24} = 1$ and since there is no more demand at $M_4$

$$x_{34} = 0$$

- We delete column 4 from any further consideration since all the demand at $M_4$ is met.
The remaining supply at $W_2$ is

$$7 - 1 = 6,$$

which is the value for the modified supply at $W_2$

We repeat these steps until we find the values of the $(m + n - 1)$ nonzero basic variables to obtain a basic feasible solution

In the reduced tableau,
## APPLICATION OF THE LEAST – COST RULE

<table>
<thead>
<tr>
<th></th>
<th>Market ( j )</th>
<th>( M_1 )</th>
<th>( M_2 )</th>
<th>( M_3 )</th>
<th>( a_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w/h ) ( i )</td>
<td>( W_2 )</td>
<td>10</td>
<td>8</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>( W_3 )</td>
<td>7</td>
<td>6</td>
<td>0</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>( b_j )</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: The least cost rule is applied by selecting the minimum cost for each market, which is highlighted with a green circle.
APPLICATION OF THE LEAST – COST RULE

- pick $x_{23}$ to enter the basis as the next basic variable

- set

$$x_{23} = \min \{ 6, 4 \} = 4$$

and set $x_{33} = 0$

- eliminate column 3 and reduce the supply at $W_2$ to

$$6 - 4 = 2$$

- For the reduced tableau
APPLICATION OF THE LEAST – COST RULE

<table>
<thead>
<tr>
<th>market ( j )</th>
<th>( M_1 )</th>
<th>( M_2 )</th>
<th>( a_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W_2 )</td>
<td>10</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>( W_3 )</td>
<td>7</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>( b_j )</td>
<td>4</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>
APPLICATION OF THE LEAST–COST RULE

○ pick $x_{32}$ to enter the basis

○ set

$$x_{32} = \min \{3, 5\} = 3$$

and set $x_{22} = 0$

○ eliminate column 2 in the reduced tableau and reduce the supply at $W_3$ to

$$5 - 3 = 2$$

☐ The last reduced tableau is
### APPLICATION OF THE LEAST – COST RULE

<table>
<thead>
<tr>
<th>market $j$</th>
<th>$M_1$</th>
<th>$a_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_2$</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>$W_3$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$b_j$</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>
APPLICATION OF THE LEAST–COST RULE

- Pick $x_{31}$ to enter the basis
- Set

$$x_{31} = \min \{2, 4\} = 2$$

- Reduce the demand at $M_1$ to

$$4 - 2 = 2$$

- The value of

$$x_{21} = 2$$

is obtained by default
# INITIAL BASIC FEASIBLE SOLUTION

<table>
<thead>
<tr>
<th></th>
<th>( M_1 )</th>
<th>( M_2 )</th>
<th>( M_3 )</th>
<th>( M_4 )</th>
<th>( a_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W_1 )</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>( W_2 )</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>( W_3 )</td>
<td>2</td>
<td>3</td>
<td></td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>( b_j )</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>
APPLICATION OF THE LEAST – COST RULE

The feasible solution involves only the basic variables and results in shipment costs of

\[ \sum_{i=1}^{3} \sum_{j=1}^{4} c_{ij} x_{ij} = 1 \cdot 3 + 4 \cdot 1 + 5 \cdot 4 + 6 \cdot 3 + 7 \cdot 2 + 10 \cdot 2 = 79 \]
The primal problem is

\[
\begin{align*}
\text{min } Z &= \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\text{s.t.} \\
u_i &\leftrightarrow \sum_{j=1}^{n} x_{ij} = a_i \quad i = 1, \ldots, m \\
v_j &\leftrightarrow \sum_{i=1}^{m} x_{ij} = b_j \quad j = 1, \ldots, n \\
x_{ij} &\geq 0
\end{align*}
\]

(P)
The dual problem is

$$\max W = \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j$$

s.t.

$$x_{ij} \iff u_i + v_j \leq c_{ij} \quad i = 1, \ldots, m$$

$$j = 1, \ldots, n$$

$$u_i, v_j \text{ are unrestricted in sign}$$
The complementary slackness conditions for \((D)\) are:

\[
x^*_{ij} [u^*_i + v^*_j - c_{ij}] = 0
\]

\(i = 1, \ldots, m\)

\(j = 1, \ldots, n\)

Due to the equalities in \((P)\), the complementary slackness conditions in \((P)\) cannot provide any useful information.
The complementary slackness conditions obtain

\[ x_{ij}^* > 0 \implies u_i^* + v_j^* = c_{ij} \]

\[ u_i^* + v_j^* < c_{ij} \implies x_{ij}^* = 0 \]

We make use of these complementary slackness conditions to develop the so-called \( u - v \) method for solving the standard transportation problem.
The \( u - v \) method starts with a basic feasible solution for the primal problem, determines the corresponding dual variables (as if the basic feasible solution were optimal) and uses the duals to determine the adjacent basic feasible solution; the process continues until the optimality conditions are satisfied.
For a basic feasible solution, we find the dual variable $u_i$ and $v_j$ using the complementary slackness conditions

$$u_i + v_j = c_{ij} \quad \forall \text{ basic } x_{ij}$$

with $u_i$ and $v_j$ being unrestricted in sign.
THE $u - v$ METHOD

- We compute

$$\tilde{c}_{ij} = c_{ij} - (u_i + v_j) \quad \forall \text{ nonbasic } x_{ij}$$

- This step is the analogue of computing $\tilde{c}^T$ in the simplex tableau approach (relative cost reduction vector)

- The complementary-slackness-based optimality test is performed:

$$\text{if } \tilde{c}_{ij} \geq 0 \quad \forall \text{ nonbasic } x_{ij} \left[ x_{ij} = 0 \right], \text{ then the basic feasible solution is optimal}$$
Otherwise, we consider all nonbasic variables $x_{\overline{p}\overline{q}}$ that satisfy

$$\tilde{c}_{\overline{p}\overline{q}} = c_{\overline{p}\overline{q}} - (u_{\overline{p}} + v_{\overline{q}}) < 0$$

and determine

$$\tilde{c}_{pq} = \min \left\{ \tilde{c}_{\overline{p}\overline{q}} \right\}$$

where $\overline{p}\overline{q} \ni x_{\overline{p}\overline{q}}$ is nonbasic and $\tilde{c}_{\overline{p}\overline{q}} < 0$

We, then, select $x_{pq}$ to become the next basic variable and repeat the process for this new basic feasible solution and continue the process until the optimality conditions are met.
We apply the $u - v$ scheme to the example previously discussed.

The basic step from the dual formulation is to require

$$u_i + v_j = c_{ij} \quad \forall \text{ basic } x_{ij}$$
We start with the basic feasible solution and apply the complementary slackness conditions

\[
\begin{align*}
   u_1 + v_4 &= 1 = c_{14} \\
   u_2 + v_4 &= 4 = c_{24} \\
   u_2 + v_3 &= 5 = c_{23} \\
   u_3 + v_2 &= 6 = c_{32} \\
   u_3 + v_1 &= 7 = c_{31} \\
   u_2 + v_1 &= 10 = c_{21}
\end{align*}
\]

We have 6 equations in 7 unknowns and so there is an infinite number of solutions
Arbitrarily, we set

\[ v_4 = 0 \]

and solve the equations above to obtain

\[ u_1 = 1 \]
\[ u_2 = 4 \]
\[ v_3 = 1 \]
\[ v_1 = 6 \]
\[ u_3 = 1 \]
\[ v_2 = 5 \]
STP  NUMERICAL EXAMPLE

☐ The $\tilde{c}_{ij}$ for the nonbasic variables are

$x_{11}: \quad \tilde{c}_{11} = c_{11} - (u_1 + v_1) = 2 - (1+6) = -5$

$x_{12}: \quad \tilde{c}_{12} = c_{12} - (u_1 + v_2) = 2 - (1+5) = -4$

$x_{13}: \quad \tilde{c}_{13} = c_{13} - (u_1 + v_3) = 2 - (1+1) = 0$

$x_{34}: \quad \tilde{c}_{34} = c_{34} - (u_3 + v_4) = 8 - (1+0) = 7$

$x_{33}: \quad \tilde{c}_{33} = c_{33} - (u_3 + v_3) = 6 - (1+1) = 4$
We determine

\[
\tilde{c}_{pq} = \min_{pq \in X_{pq}} = \tilde{c}_{11} = -5
\]

and consequently we pick the nonbasic variable \( x_{11} \) to enter the basis

We determine the maximal value of \( x_{11} \) and set

\( x_{11} = \theta \) and make use of the tableau
## STP Numerical Example

<table>
<thead>
<tr>
<th>market $j$</th>
<th>$M_1$</th>
<th>$M_2$</th>
<th>$M_3$</th>
<th>$M_4$</th>
<th>$\alpha_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_1$</td>
<td>$\theta$</td>
<td></td>
<td></td>
<td>$3 - \theta$</td>
<td>3</td>
</tr>
<tr>
<td>$W_2$</td>
<td>$2 - \theta$</td>
<td>4</td>
<td>1 + $\theta$</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>$W_3$</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>$b_j$</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>
Therefore,

$$\theta = \min \{ 2, 3 \} = 2$$

Consequently, $x_{21}$ becomes $\theta$ and leaves the basis.

We obtain the basic feasible solution

$$x_{14} = 1, \ x_{11} = 2, \ x_{31} = 2, \ x_{32} = 3, \ x_{23} = 4, \ x_{24} = 3$$

and repeat to solve the $u - v$ problem for this new basic feasible solution.
### STP NUMERICAL EXAMPLE

<table>
<thead>
<tr>
<th>$w/h$ $i$</th>
<th>$u_1 = 0$</th>
<th>$u_2 = 3$</th>
<th>$u_3 = 5$</th>
<th>$a_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1 = 2$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$v_2 = 1$</td>
<td></td>
<td>4</td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>$v_3 = 2$</td>
<td></td>
<td></td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$v_4 = 1$</td>
<td></td>
<td></td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>$b_j$</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>
The complementary slackness conditions of the nonzero valued basic variables obtain:

\[ u_1 + v_1 = c_{11} = 2 \]
\[ u_1 + v_4 = c_{14} = 1 \]
\[ u_2 + v_3 = c_{23} = 5 \]
\[ u_2 + v_4 = c_{24} = 4 \]
\[ u_3 + v_1 = c_{31} = 7 \]
\[ u_3 + v_2 = c_{32} = 6 \]
STP NUMERICAL EXAMPLE

We set

\[ u_1 = 0 \]

and therefore

\[ v_3 = 2 \]
\[ v_1 = 2 \]
\[ u_3 = 5 \]
\[ u_3 = 5 \]
\[ v_2 = 1 \]
\[ v_2 = 0 \]

We compute \( \tilde{c}_{ij} \) for each nonbasic variable \( x_{ij} \)
### STP NUMERICAL EXAMPLE

<table>
<thead>
<tr>
<th>( \tilde{c}_{12} )</th>
<th>= ( c_{12} - (u_1 + v_2) )</th>
<th>= 2 - (0 + 1)</th>
<th>= 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{c}_{13} )</td>
<td>= ( c_{13} - (u_1 + v_3) )</td>
<td>= 2 - (0 + 2)</td>
<td>= 0</td>
</tr>
<tr>
<td>( \tilde{c}_{21} )</td>
<td>= ( c_{21} - (u_2 + v_1) )</td>
<td>= 10 - (3 + 2)</td>
<td>= 5</td>
</tr>
<tr>
<td>( \tilde{c}_{22} )</td>
<td>= ( c_{22} - (u_2 + v_2) )</td>
<td>= 8 - (3 + 1)</td>
<td>= 4</td>
</tr>
<tr>
<td>( \tilde{c}_{33} )</td>
<td>= ( c_{33} - (u_3 + v_3) )</td>
<td>= 6 - (5 + 2)</td>
<td>= -1</td>
</tr>
<tr>
<td>( \tilde{c}_{34} )</td>
<td>= ( c_{34} - (u_3 + v_4) )</td>
<td>= 8 - (5 + 1)</td>
<td>= 2</td>
</tr>
</tbody>
</table>

- only possible improvement

- We introduce \( x_{33} \) as a *basic variable* and determine its *nonnegative value* \( \theta \) from the tableau
### STP NUMERICAL EXAMPLE

<table>
<thead>
<tr>
<th>market $j$</th>
<th>$M_1$</th>
<th>$M_2$</th>
<th>$M_3$</th>
<th>$M_4$</th>
<th>$a_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_1$</td>
<td>$2 + \theta$</td>
<td></td>
<td>$1 - \theta$</td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>$W_2$</td>
<td></td>
<td>$4 - \theta$</td>
<td>$3 + \theta$</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>$W_3$</td>
<td>$2 - \theta$</td>
<td>3</td>
<td>$\theta$</td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>$b_j$</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>
The limiting value of $q$ is

$$\theta = \min \{ 2, 4, 1 \} = 1$$

Consequently, $x_{14}$ leaves the basis and $x_{33}$ enters the basis with the value 1

We obtain the adjacent basic feasible solution in
**STP NUMERICAL EXAMPLE**

<table>
<thead>
<tr>
<th></th>
<th>$v_1 = 2$</th>
<th>$v_2 = 1$</th>
<th>$v_3 = 1$</th>
<th>$v_4 = 0$</th>
<th>$a_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_1 = 0$</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$u_2 = 4$</td>
<td>10</td>
<td>8</td>
<td>5</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>$u_3 = 5$</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>$b_j$</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>
We evaluate $\tilde{c}_{ij}$ for each nonbasic variable; $\tilde{c}_{ij} \geq 0$ and so we have an optimal solution with

<table>
<thead>
<tr>
<th>Shipping</th>
<th>From</th>
<th>To</th>
<th>With costs</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$W_1$</td>
<td>$M_1$</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>$W_3$</td>
<td>$M_1$</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>$W_3$</td>
<td>$M_2$</td>
<td>18</td>
</tr>
<tr>
<td>1</td>
<td>$W_3$</td>
<td>$M_3$</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>$W_2$</td>
<td>$M_3$</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>$W_2$</td>
<td>$M_4$</td>
<td>16</td>
</tr>
</tbody>
</table>

and resulting in the least total costs of 68
We consider an electric utility system in which 3 power plants are used to supply the electricity demand of 4 cities.

The supplies available from the 3 plants are given.

The demands of the 4 cities are specified.

The costs of supply per $10^6 \text{kWh}$ are given.
## ELECTRICITY COSTS

<table>
<thead>
<tr>
<th>Plant</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Supplies $(10^6 \text{ kWh})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>6</td>
<td>10</td>
<td>9</td>
<td>35</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>12</td>
<td>13</td>
<td>7</td>
<td>50</td>
</tr>
<tr>
<td>3</td>
<td>14</td>
<td>9</td>
<td>16</td>
<td>5</td>
<td>40</td>
</tr>
</tbody>
</table>

| Demands $(10^6 \text{ kWh})$ | 45 | 20 | 30 | 30 | 125 |

From plant 1 to city 1:
- Cost: 8

From plant 2 to city 2:
- Cost: 12

From plant 3 to city 3:
- Cost: 16

Total costs:
- Plant 1 to city 1: 8
- Plant 2 to city 2: 12
- Plant 3 to city 3: 16
- Supplies: 35 + 50 + 40 = 125

Total demands: 45 + 20 + 30 + 30 = 125

Total costs match total demands.
# ELECTRICITY COSTS

## Balanced Transportation Problem

<table>
<thead>
<tr>
<th></th>
<th>city</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>from</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>14</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>45</td>
<td>20</td>
</tr>
</tbody>
</table>

**Supplies** (10^6 kWh)

- City 1: 35
- City 2: 50
- City 3: 40
- City 4: 125

**Demands** (10^6 kWh)

- City 1: 45
- City 2: 20
- City 3: 30
- City 4: 30

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We note that
\[ \sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j \]
and so we have a balanced transportation problem.

We find a \textit{basic feasible solution} using the least-cost rule.
ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

<table>
<thead>
<tr>
<th>from</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>supplies (10^6 kWh)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>6</td>
<td>10</td>
<td>0</td>
<td>35</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>12</td>
<td>13</td>
<td>0</td>
<td>50</td>
</tr>
<tr>
<td>3</td>
<td>14</td>
<td>9</td>
<td>16</td>
<td>30</td>
<td>10</td>
</tr>
</tbody>
</table>

| demands (10^6 kWh) | 45 | 20 | 30 | 30 | 125 |

plant

to

city
And we set

\[ x_{34} = 30 \]
\[ x_{14} = 0 \]
\[ x_{24} = 0 \]

We compute the remaining supply at plant 3 and remove column corresponding to city 4 from further consideration.

We continue with the reduced system.
ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

<table>
<thead>
<tr>
<th>from</th>
<th>plant</th>
<th>to</th>
<th>city</th>
<th>supplies (10^6 kWh)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td>15</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>16</td>
<td></td>
</tr>
</tbody>
</table>

demands (10^6 kWh)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>45</td>
<td>0</td>
<td>20</td>
<td>30</td>
</tr>
</tbody>
</table>
ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

and so we set

\[ x_{12} = 20 \]
\[ x_{22} = 0 \]
\[ x_{32} = 0 \]

- We recompute the supply remaining at plant 1 and remove column corresponding to city 2

- The new reduced system obtains
## ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

<table>
<thead>
<tr>
<th>from</th>
<th>to</th>
<th>city</th>
<th>supplies (10^6 kWh)</th>
</tr>
</thead>
<tbody>
<tr>
<td>plant</td>
<td>1</td>
<td>1</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>demands (10^6 kWh)</td>
<td>30</td>
<td>30</td>
<td>10</td>
</tr>
</tbody>
</table>
and therefore we set

\[ x_{11} = 15 \]
\[ x_{13} = 0 \]

and remove the row corresponding to plant 1 from further consideration since its supply is exhausted.

The operation is repeated on the reduced system.
### ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

<table>
<thead>
<tr>
<th>from</th>
<th>plant</th>
<th>to</th>
<th>city</th>
<th>supplies (10^6 kWh)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
<td></td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>30</td>
<td>9</td>
<td>13</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>14</td>
<td>16</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>30</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**demands (10^6 kWh):**
- From plant 2 to city 1: 30
- From plant 3 to city 3: 30
and therefore we set

\[ x_{21} = 30 \]

\[ x_{31} = 0 \]

and remove the column corresponding to city 1 from further consideration

- We are finally left with
**ELECTRICITY ALLOCATION EXAMPLE: SOLUTION**

<table>
<thead>
<tr>
<th>from</th>
<th>to</th>
<th>city</th>
<th>supplies (10^6 kWh)</th>
</tr>
</thead>
<tbody>
<tr>
<td>plant</td>
<td>2</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td></td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>demands</td>
<td></td>
<td>30</td>
<td></td>
</tr>
</tbody>
</table>

Note: The supplies are given as 10^6 kWh, and the demands are also in terms of 10^6 kWh.
ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

which allows us to set

\[ x_{23} = 20 \]
\[ x_{33} = 10 \]

- The basic feasible solution has the costs

\[ Z = 30 \cdot 5 + 20 \cdot 6 + 15 \cdot 8 + 30 \cdot 9 + 20 \cdot 13 + 10 \cdot 16 = 1,080 \]

- We improve this solution by using the \( u - v \) scheme

- The first tableau corresponding to the initial basic feasible solution is:
### ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>supplies (10^6 kWh)</th>
</tr>
</thead>
<tbody>
<tr>
<td>from</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>35</td>
</tr>
<tr>
<td>plant</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>50</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>40</td>
</tr>
<tr>
<td>1</td>
<td>15</td>
<td>20</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td></td>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>10</td>
<td>30</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>6</td>
<td>16</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

demands (10^6 kWh):

- From plant 1: 15 kWh to city 1, 20 kWh to city 2
- From plant 2: 30 kWh to city 1, 20 kWh to city 3
- From plant 3: 10 kWh to city 2, 30 kWh to city 3

Supplies:

- Total from plant 1 to city 1 and 2: 35 kWh
- Total from plant 2 to city 2 and 3: 50 kWh
- Total from plant 3 to city 2 and 3: 40 kWh

Demands:

- Total demand for cities 1 to 4: 45, 20, 30, 30 kWh

This example shows the allocation of electricity supplies to meet city demands efficiently.
STP NUMERICAL EXAMPLE

- We compute, the possible improvements at each nonbasic variable:

\[
\begin{align*}
\tilde{c}_{31} &= c_{31} - (u_3 + v_1) = 14 - (4 + 8) = 2 \\
\tilde{c}_{22} &= c_{22} - (u_2 + v_2) = 12 - (1 + 6) = 5 \\
\tilde{c}_{32} &= c_{32} - (u_3 + v_2) = 9 - (4 + 6) = -1 \\
\tilde{c}_{13} &= c_{13} - (u_1 + v_3) = 10 - (0 + 12) = -2 \\
\tilde{c}_{14} &= c_{14} - (u_1 + v_4) = 9 - (0 + 1) = 8 \\
\tilde{c}_{24} &= c_{24} - (u_2 + v_4) = 7 - (1 + 1) = 5
\end{align*}
\]

Improvement possible

Better improvement
We bring $x_{13}$ into the basis and determine the value of $q$ using the tableau structure.

From the tableau we conclude that

$$\theta = \min \{ 15, 20 \} = 15$$

and therefore $x_{11}$ leaves the basis to determine the adjacent basic feasible solution given in the table.
### STP Numerical Example

<table>
<thead>
<tr>
<th>cities</th>
<th>plants</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>$a_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>15-θ</td>
<td>20</td>
<td>θ</td>
<td></td>
<td>35</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>30+θ</td>
<td></td>
<td>20-θ</td>
<td></td>
<td>50</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>10</td>
<td>30</td>
<td></td>
<td>40</td>
</tr>
<tr>
<td>$b_j$</td>
<td>45</td>
<td>20</td>
<td>30</td>
<td>30</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
STP NUMERICAL EXAMPLE

The adjacent basic feasible solution is

\[ x_{21} = 45, \quad x_{12} = 20, \quad x_{13} = 15, \quad x_{23} = 5, \quad x_{33} = 10, \quad x_{34} = 30 \]

and the new value of \( Z \) is

\[ Z = 20 \cdot 6 + 15 \cdot 10 + 45 \cdot 9 + 5 \cdot 13 + 10 \cdot 16 + 30 \cdot 5 \]

\[ = 1050 < 1080 \]

We again pursue a \( u - v \) improvement strategy by starting with the tableau
## STP NUMERICAL EXAMPLE

<table>
<thead>
<tr>
<th>cities</th>
<th>( \nu_1 = 6 )</th>
<th>( \nu_2 = 6 )</th>
<th>( \nu_3 = 10 )</th>
<th>( \nu_4 = -1 )</th>
<th>supplies</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u_1 = 0 )</td>
<td></td>
<td>20</td>
<td>15</td>
<td></td>
<td>35</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6</td>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( u_2 = 3 )</td>
<td>45</td>
<td>5</td>
<td></td>
<td></td>
<td>50</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>13</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( u_3 = 6 )</td>
<td></td>
<td>10</td>
<td>30</td>
<td></td>
<td>40</td>
</tr>
<tr>
<td></td>
<td></td>
<td>16</td>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( u_3 = 6 )</td>
<td></td>
<td>10</td>
<td>30</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>16</td>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>demands</td>
<td>45</td>
<td>20</td>
<td>30</td>
<td>30</td>
<td></td>
</tr>
</tbody>
</table>
The complementary slackness conditions obtain the possible improvements

\[ \tilde{c}_{11} = c_{11} - (u_1 + v_1) = 8 - (\theta + 6) = 2 \]

\[ \tilde{c}_{31} = c_{31} - (u_3 + v_1) = 14 - (6 + 6) = 2 \]

\[ \tilde{c}_{22} = c_{22} - (u_2 + v_2) = 12 - (3 + 6) = 3 \]

\[ \tilde{c}_{32} = c_{32} - (u_3 + v_2) = 9 - (6 + 6) = -3 \]

\[ \tilde{c}_{14} = c_{14} - (u_1 + v_4) = 9 - (\theta - 1) = 10 \]

\[ \tilde{c}_{24} = c_{24} - (u_2 + v_4) = 7 - (3 - 1) = 5 \]

only possible improvement

We bring \( x_{32} \) into the basis and with its value \( \theta \) determined from
### STP NUMERICAL EXAMPLE

<table>
<thead>
<tr>
<th>cities</th>
<th>plants</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>( \alpha_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>20-( \theta )</td>
<td>15+( \theta )</td>
<td></td>
<td></td>
<td>35</td>
</tr>
<tr>
<td>2</td>
<td>45</td>
<td></td>
<td>5</td>
<td></td>
<td></td>
<td>50</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>( \theta )</td>
<td>10-( \theta )</td>
<td>30</td>
<td></td>
<td>40</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td>30</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>b_j</td>
<td>45</td>
<td>20</td>
<td>30</td>
<td>30</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
STP NUMERICAL EXAMPLE

and so

\[ \theta = \min \{ 10, 20 \} = 10 \]

The adjacent basic feasible solution is, then,

\[ x_{21} = 45 \quad x_{12} = 10 \quad x_{32} = 10 \]
\[ x_{13} = 25 \quad x_{23} = 5 \quad x_{34} = 30 \]

and the value of \( Z \) becomes

\[ Z = 45 \cdot 9 + 10 \cdot 6 + 10 \cdot 9 + 25 \cdot 10 + 5 \cdot 13 + 30 \cdot 5 = 1,020 \]

You are asked to prove, using complementary slackness conditions, that this is the optimum
NONSTANDARD TRANSPORTATION PROBLEM

- The nonstandard transportation problem arises when supply and demand are unbalanced: either supply exceeds demand or vice versa.
- We solve by transforming the nonstandard problem into a standard one.
- The approach is to create a *fictitious* entity – a market to absorb the surplus supply or a warehouse for the supply deficit – and solve the problem with the fictitious entity as a balanced problem.
For the case

\[ \sum_{i=1}^{m} a_i > \sum_{i=1}^{n} b_j \]

we create the fictitious market \( M_{n+1} \) to absorb all the excess supply \( \left( \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j \right) \); we set \( c_{i,n+1} = 0 \), \( \forall \ i = 1, 2, \ldots, m \) since \( M_{n+1} \) is fictitious

The problem is then in standard form with \( j = 1, 2, \ldots, n, n+1 \), for the augmented number of markets
For the case

\[ \sum_{j=1}^{n} b_{j} > \sum_{i=1}^{m} a_{i} \]

the problem is not, in effect, feasible since all the demands cannot be met and therefore the least-cost shipping schedule is that which will supply as much as possible of the demands of the markets at the lowest cost.
For the excess demand case, we introduce the fictitious warehouse $W_{m+1}$ to supply the shortage

$$\left[ \sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i \right]$$

and we set $c_{m+1,j} = 0, \ j = 1, 2, \ldots, n$

The problem is in standard form with $i = 1, \ldots, m+1$ (number of warehouses augmented by 1)
Note that the variable $x_{m+1,j}$ is the shortage at market $j$ and is the shortfall in the demand $b_j$ experienced by each market $M_j$ due to inadequacy of the supplies $j = 1, 2, \ldots, n$

For each market $j$, $x_{m+1,j}$ provides the measure of the infeasibility of the problem.
EXAMPLE: CANNING OPERATIONS SCHEDULING

This problem is concerned with the scheduling the purchases of 2 plants – $A$ and $B$ – of the raw supplies from 3 growers with given availability / price.

<table>
<thead>
<tr>
<th>grower</th>
<th>availability (ton)</th>
<th>price ( $/ton )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Smith</td>
<td>200</td>
<td>10</td>
</tr>
<tr>
<td>Jones</td>
<td>300</td>
<td>9</td>
</tr>
<tr>
<td>Richard</td>
<td>400</td>
<td>8</td>
</tr>
</tbody>
</table>
The shipping costs in $/ton are given by

<table>
<thead>
<tr>
<th></th>
<th>to</th>
<th>plant</th>
</tr>
</thead>
<tbody>
<tr>
<td>from</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Smith</td>
<td>2</td>
<td>2.5</td>
</tr>
<tr>
<td>Jones</td>
<td>1</td>
<td>1.5</td>
</tr>
<tr>
<td>Richard</td>
<td>5</td>
<td>3</td>
</tr>
</tbody>
</table>
### EXAMPLE: CANNING OPERATIONS SCHEDULING

The plants’ capacity limits and labor costs are

<table>
<thead>
<tr>
<th>plant</th>
<th>$A$</th>
<th>$B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>capacity (ton)</td>
<td>450</td>
<td>550</td>
</tr>
<tr>
<td>labor costs ($/ton)</td>
<td>25</td>
<td>20</td>
</tr>
</tbody>
</table>
EXAMPLE: CANNING OPERATIONS SCHEDULING

- The competitive selling price for canned goods is $50/ton and the company can sell all it produces.
- The problem is to determine the purchase schedule that produces the maximum profits.
- Note that this is an unbalanced problem since
  
  \[
  supply = 200 + 300 + 400 = 900 \text{ tons}
  \]
  \[
  demand = 450 + 550 = 1000 \text{ tons} > 900 \text{ tons}
  \]
- The decision variables are the amounts bought from each grower and shipped to each plant.
The objective is formulated as

$$\max Z = \begin{bmatrix} 50 - 25 - 10 - 2 \\ 13 \end{bmatrix} x_{SA} + \begin{bmatrix} 50 - 25 - 9 - 1 \\ 15 \end{bmatrix} x_{JA} + \begin{bmatrix} 50 - 25 - 8 - 5 \\ 12 \end{bmatrix} x_{RA} + \begin{bmatrix} 50 - 20 - 10 - 2.5 \\ 17.5 \end{bmatrix} x_{SB} + \begin{bmatrix} 50 - 20 - 9 - 1.5 \\ 19.5 \end{bmatrix} x_{JB} + \begin{bmatrix} 50 - 20 - 8 - 3 \\ 19 \end{bmatrix} x_{RB}$$
EXAMPLE: CANNING OPERATIONS SCHEDULING

- The supply constraints are

\[ x_{SA} + x_{SB} \leq 200 \]
\[ x_{JA} + x_{JB} \leq 300 \]
\[ x_{RA} + x_{RB} \leq 400 \]

- The demand constraints are

\[ x_{SA} + x_{JA} + x_{RA} \leq 450 \]
\[ x_{SB} + x_{JA} + x_{RB} \leq 550 \]
Clearly, all decision variables are nonnegative.

The unbalanced nature of the problem requires the introduction of a fictitious grower $F$, who is able to supply 100 tons of the supply shortage; the addition of $F$ allows the nonstandard problem to be stated as a standard transportation problem.

We set up the $STP$ tableau.
### EXAMPLE: CANNING OPERATIONS SCHEDULING

<table>
<thead>
<tr>
<th>grower $i$</th>
<th>$A$</th>
<th>$B$</th>
<th>supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>13</td>
<td>17.5</td>
<td>200</td>
</tr>
<tr>
<td>$J$</td>
<td>15</td>
<td>19.5</td>
<td>300</td>
</tr>
<tr>
<td>$R$</td>
<td>12</td>
<td>19</td>
<td>400</td>
</tr>
<tr>
<td>$F$</td>
<td>0</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td><strong>demand</strong></td>
<td><strong>450</strong></td>
<td><strong>550</strong></td>
<td><strong>1,000</strong></td>
</tr>
</tbody>
</table>
EXAMPLE: CANNING OPERATIONS SCHEDULING

- In this problem, the objective is a maximization rather than a minimization.

- We therefore recast the “mechanics” of the $u - v$ scheme for the maximization problem.

- As a homework exercise, show that the duality complementary slackness conditions allow us to change the $u - v$ algorithm in the following way:
EXAMPLE: CANNING OPERATIONS SCHEDULING

- The selection of the nonbasic variable $x_{ij}$ to enter the basis is from those $x_{ij}$ whose corresponding

$$c_{ij} > u_i + v_j$$

and we focus on and evaluate all $\tilde{c}_{ij} > 0$ for which $x_{ij}$ is a candidate to enter the basis

- We pick $x_{pq}$ corresponding to

$$\tilde{c}_{pq} = \max \left\{ \tilde{c}_{\overline{p} \overline{q}} \right\}$$

where $\overline{p} \overline{q} \ni x_{\overline{p} \overline{q}}$ is nonbasic and $\tilde{c}_{\overline{p} \overline{q}} > 0$
## EXAMPLE SOLUTION

<table>
<thead>
<tr>
<th>grower i</th>
<th>plant j</th>
<th>A</th>
<th>B</th>
<th>supply</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S</td>
<td></td>
<td>200</td>
<td>0</td>
<td>200</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>J</td>
<td></td>
<td>250</td>
<td>50</td>
<td>300</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
<td>0</td>
<td>400</td>
<td>400</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td></td>
<td>0</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>demand</td>
<td></td>
<td>450</td>
<td>550</td>
<td></td>
</tr>
</tbody>
</table>
EXAMPLE SOLUTION

We construct the $u - v$ relations for this solution

\[
\begin{align*}
    u_1 + v_1 &= 13 \\
    u_2 + v_1 &= 15 \\
    u_3 + v_2 &= 19 \\
    u_4 + v_2 &= 0
\end{align*}
\]

We arbitrarily set $u_1 = 0$ and compute

\[
\begin{align*}
    v_1 &= 13, \
    u_2 &= 2, \
    v_2 &= 17.5, \
    u_3 &= 1.5, \
    u_4 &= -17.5
\end{align*}
\]
EXAMPLE SOLUTION

We evaluate the $\tilde{c}_{ij}$ corresponding to the nonbasic variables

$$\tilde{c}_{31} = c_{31} - (u_3 + v_1) = 12 - (1.5 + 13) = -2.5$$

$$\tilde{c}_{41} = c_{41} - (u_4 + v_1) = 0 - (-17.5 + 13) = 4.5$$

$$\tilde{c}_{12} = c_{12} - (u_1 + v_2) = 17.5 - (0 + 17.5) = 0$$

single possible improvement

Thus, $x_{41}$ enters the basis and we determine $\theta$
### EXAMPLE SOLUTION

#### Plant to Grower Distribution

<table>
<thead>
<tr>
<th>Plant \ Grower</th>
<th>A</th>
<th>B</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>200</td>
<td>13</td>
<td>200</td>
</tr>
<tr>
<td>J</td>
<td>250 - θ</td>
<td>50 + θ</td>
<td>300</td>
</tr>
<tr>
<td>R</td>
<td></td>
<td>400</td>
<td>400</td>
</tr>
<tr>
<td>F</td>
<td>θ</td>
<td>100 - θ</td>
<td>100</td>
</tr>
</tbody>
</table>

**Demand**

| Demand | 450 | 550 |
EXAMPLE SOLUTION

- It follows that

\[ \theta = \min \{ 250, 100 \} = 100 \]

and so the adjacent basic feasible solution is

\[ x_{11} = 200, \ x_{21} = 150, \ x_{41} = 100, \ x_{22} = 150, \ x_{32} = 400 \]

- We repeat the \( u - v \) procedure with the new basic variables and solve
EXAMPLE SOLUTION

\[ u_1 + v_1 = 13 \]
\[ u_2 + v_1 = 15 \]
\[ u_2 + v_2 = 19.5 \]
\[ u_3 + v_2 = 19 \]
\[ u_4 + v_1 = 0 \]

We solve by arbitrarily setting \( u_1 = 0 \) and obtain

\[ v_1 = 13, \ u_2 = 2, \ v_2 = 17.5, \ u_3 = 1.5, \ u_4 = -13 \]
EXAMPLE SOLUTION

We compute the $\tilde{c}_{ij}$ for the nonbasic variables

\[ \tilde{c}_{12} = 17.5 - (0 + 17.5) = 0 \]

\[ \tilde{c}_{31} = 12 - (1.5 + 13) = -2.5 \]

\[ \tilde{c}_{42} = 0 - (-13 + 17.5) = -4.5 \]
EXAMPLE SOLUTION

Since each \( \delta_{ij} \) is \( \leq \delta \), no improvement in the maximization is possible and so the maximum profits are

\[
Z = (200)13 + (150)15 + (100)0 + (150)19.5 + (400)19
\]

\[
= 15,375 \, \$\]
The problem is concerned with the weekly production scheduling over a 4-week period.

- Production costs for each item:
  - First two weeks: $10
  - Last two weeks: $15

- Demands that need to be met are:

<table>
<thead>
<tr>
<th>Week</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Demand</td>
<td>300</td>
<td>700</td>
<td>900</td>
<td>800</td>
</tr>
</tbody>
</table>
SCHEDULING PROBLEM AS A STANDARD TRANSPORTATION PROBLEM

- weekly plant capacity is 700
- overtime is possible for weeks 2 and 3 with
  - the production of additional 200 units
  - additional cost per unit of $5
- $3 for weekly storage of unsold production
- the objective is to minimize the total costs for the 4-week schedule

The decision variables are

\[ x_{ij} = \text{production in week } i \text{ for use in week } j \text{ market} \]
SCHEDULING PROBLEM AS A STANDARD TRANSPORTATION PROBLEM

<table>
<thead>
<tr>
<th>demand wk.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>F</th>
<th>supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>production wk.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>700</td>
</tr>
<tr>
<td>2</td>
<td>normal</td>
<td>M</td>
<td>10</td>
<td>13</td>
<td>16</td>
<td>19</td>
</tr>
<tr>
<td></td>
<td>o/t</td>
<td>M</td>
<td>15</td>
<td>18</td>
<td>21</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>normal</td>
<td>M</td>
<td>M</td>
<td>15</td>
<td>18</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>o/t</td>
<td>M</td>
<td>M</td>
<td>20</td>
<td>23</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>demand</td>
<td>300</td>
<td>700</td>
<td>900</td>
<td>800</td>
<td>500</td>
<td></td>
</tr>
</tbody>
</table>
### Scheduling Problem as a Standard Transportation Problem

<table>
<thead>
<tr>
<th>Demand wk.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>F</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>13</td>
<td>16</td>
<td>19</td>
<td>0</td>
<td>700</td>
</tr>
<tr>
<td>2</td>
<td>M</td>
<td>10</td>
<td>13</td>
<td>16</td>
<td>0</td>
<td>700</td>
</tr>
<tr>
<td>3</td>
<td>M</td>
<td>M</td>
<td>15</td>
<td>18</td>
<td>0</td>
<td>700</td>
</tr>
<tr>
<td>4</td>
<td>M</td>
<td>M</td>
<td>M</td>
<td>15</td>
<td>0</td>
<td>700</td>
</tr>
</tbody>
</table>

- **M** is a very large number.
- **3,200** is the available production for week 2.
- **2,700** is the demand for week 4.
- **3,200 – 2,700** is the difference between the available production and demand for week 4.
ASSIGNMENT PROBLEM

We are given

- $n$ machines $M_1, M_2, \ldots, M_n$ ↔ $i$
- $n$ jobs $J_1, J_2, \ldots, J_n$ ↔ $j$

$c_{ij} =$ cost of doing job $j$ on machine $i$

$c_{ij} = M$ if job $j$ cannot be done on machine $i$

Each machine can only do one job and we wish to determine the optimal match, i.e., the assignment with the lowest total costs of doing each job $j$ on the $n$ available machines.
The brute force approach is simply enumeration:

consider \( n = 10 \) and there are \( 3,628,800 \) possible choices!

We can, however, introduce \textit{categorical} decision variables

\[
\begin{align*}
x_{ij} &= \begin{cases} 
1 & \text{job } j \text{ is assigned to machine } i \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]
and the problem constraints can be stated as

\[ \sum_{j=1}^{n} x_{ij} = 1 \quad \forall i \text{ each machine does exactly 1 job} \]

\[ \sum_{i=1}^{n} x_{ij} = 1 \quad \forall j \text{ each job is assigned to 1 machine} \]

\[ \Box \text{ The objective, then, is} \]

\[ \min Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \]
This assignment problem is an STP with

\[ a_i = 1 \quad \forall i \]

\[ b_j = 1 \quad \forall j \]

\[ \sum_{i=1}^{n} a_i = \sum_{j=1}^{n} b_j \]
Suppose we have \( m \) machines and \( n \) jobs with \( m \neq n \).

We may convert this into an equivalent standard assignment problem with equal number of machines and jobs.

The conversion requires the introduction of either fictitious jobs or fictitious machines.
In the case $m > n$:

we create $(m - n)$ fictitious jobs and we have $m$ machines and $n + m - n = m$ jobs; we assign the machinery costs for the fictitious goods to be $0$: note that there is no change in the objective function since a fictitious job assigned to a machine is, in effect, a machine that remains idle.
For the case $n > m$:

we create $(n - m)$ fictitious machines with machine costs of 0 and the solution obtained has the $(n - m)$ jobs that cannot be done due to lack of machines.
In principle, any assignment problem may be solved using the transportation problem technique; in practice, this approach is not practical since there exists degeneracy in every basic feasible solution.

We note that in the standard assignment problem for $m$ machines with $m = n$, there are exactly $m$ $x_{ij}$ that are 1 (nonzero) but every basic feasible solution of the transportation problem has $(2m - 1)$ basic variables of which $(m - 1)$ have the value zero.