ECE 307 – Techniques for Engineering Decisions

Lecture 2. Introduction to Linear Programming

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OUTLINE

- The nature of a programming or optimization problem
- Linear programming \((LP)\) : salient characteristics
- The \(LP\) problem formulation
- The \(LP\) problem solution
- Extensive illustrations with numerical examples
EXAMPLE 1: HIGH/LOW HEEL SHOE CHOICE PROBLEM

- A lady is headed to a party and is trying to find a pair of shoes to wear; the choice is narrowed down to two possible choices:
  - a high heel pair; and
  - a low heel pair

- The high heel shoes look more beautiful but are not as comfortable as the competing pair

- Which pair should she choose?
MODEL FORMULATION

- We first quantify our assessment along the two dimensions of *looks* and *comfort* in a table.

<table>
<thead>
<tr>
<th>aspect</th>
<th>maximum value</th>
<th>assessment</th>
<th>weight (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>high heels</td>
<td>low heels</td>
</tr>
<tr>
<td>aesthetics</td>
<td>5.0</td>
<td>4.2</td>
<td>3.6</td>
</tr>
<tr>
<td>comfort</td>
<td>5.0</td>
<td>3.5</td>
<td>4.8</td>
</tr>
</tbody>
</table>

- Next, we represent the decision in terms of two decision variables:
We formulate the objective to be the maximization of the weighted assessment

$$\max \{ 70 \% \ast \text{aesthetics} + 30 \% \ast \text{comfort} \}$$

We state the objective in terms of the defined decision variables

$$\max Z = x_H [(4.2)(0.7)+(3.5)(0.3)] + x_L [(3.6)(0.7)+(4.8)(0.3)]$$
Next, we consider the problem constraints:

- only one pair of shoes can be selected
- each decision variable is nonnegative

We express the constraints in terms of $x_H$ and $x_L$

\[
x_H + x_L = 1
\]
\[
x_H \geq 0, \quad x_L \geq 0
\]
PROBLEM STATEMENT SUMMARY

- **Decision variables:**
  
  \[ x_H = \begin{cases} 
  1 & \text{choose high} \\
  0 & \text{otherwise} 
  \end{cases} \quad x_L = \begin{cases} 
  1 & \text{choose low} \\
  0 & \text{otherwise} 
  \end{cases} \]

- **Objective function:**
  
  \[ \max Z = 3.99 x_H + 3.96 x_L \]

- **Constraints:**
  
  \[ x_H + x_L = 1 \]
  
  \[ x_H \geq 0, x_L \geq 0 \]
OPTIMAL SOLUTION

- We determine the values $x^*_H$ and $x^*_L$ which result in the value of $Z^*$ such that
  
  $Z^* = Z\left(x^*_H, x^*_L\right) \geq Z\left(x_H, x_L\right)$

  for all feasible $(x_H, x_L)$

- We call such a solution an **optimal solution**

- A **feasible** solution is one that satisfies all the constraints

- The **optimal** solution, denoted by $(x^*_H, x^*_L)$, is selected from all the **feasible** solutions to the problem so as to satisfy (†)
SOLUTION APPROACH: EXHAUSTIVE SEARCH

- We enumerate all the feasible solutions: in this problem there are only two alternatives:

\[
A : \begin{cases}
x_H = 1 \\
x_L = 0
\end{cases}
\quad B : \begin{cases}
x_H = 0 \\
x_L = 1
\end{cases}
\]

- We evaluate \( Z \) for \( A \) and \( B \) and compare

\[
Z_A = 3.99 \quad \text{and} \quad Z_B = 3.96
\]

so that \( Z_A > Z_B \) and so \( A \) is the optimal choice

- The \textit{optimal} solution is

\[
x^*_H = 1, \quad x^*_L = 0 \quad \text{and} \quad Z^* = 3.99
\]
The objective is to make the decision among the various alternatives and therefore requires the definition of the decision variables.

The solution of the “best” decision is made on the basis of the objective function and thus requires the objective function mathematical formulation.

The decision must satisfy each specified constraint and so we require the mathematical statement of the problem constraints.
The problem statement is characterized by:

- **Decision variables**: continuous valued, integer valued
- **Objective function**: linear, non-linear
- **Constraints**: linear, non-linear
PROGRAMMING PROBLEM CLASSES

- Linear/nonlinear programming
- Static/dynamic programming
- Integer programming
- Mixed programming
EXAMPLE 2: CONDUCTOR PROBLEM

- A company is producing two types of conductors for EHV transmission lines

<table>
<thead>
<tr>
<th>type</th>
<th>conductor</th>
<th>production capacity (unit/day)</th>
<th>metal needed (tons/unit)</th>
<th>profits ($/unit)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>ACSR 84/19</td>
<td>4</td>
<td>1/6</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>ACSR 18/7</td>
<td>6</td>
<td>1/9</td>
<td>5</td>
</tr>
</tbody>
</table>

- The supply department can provide up to 1 ton of metal each day
- We schedule the production so as to maximize the profits of the company
PROBLEM ANALYSIS

- Determination of the objective: to maximize the profits of the company

- Means to attain this objective: decision of how many units of product 1 and of product 2 to produce each day

- Consideration of the constraints: the daily production capacity limits, the daily metal supply limit and common sense requirements
MODEL CONSTRUCTION

- We define the decision variables to be
  \[ x_1 = \text{number of type 1 units produced per day} \]
  \[ x_2 = \text{number of type 2 units produced per day} \]

- We define the objective to be
  \[ Z = \text{profits ($/day)} \]
  \[ = 3x_1 + 5x_2 \]

- Sanity check for units of the objective function
  \[ ($/day) = ($/unit) \cdot (unit/day) \]
PROBLEM STATEMENT

- **Objective function:**
  
  \[ \text{max } Z = 3x_1 + 5x_2 \]

- **Constraints:**
  
  - capacity limits:
    \[ x_1 \leq 4 \quad x_2 \leq 6 \]
  
  - metal supply limit:
    \[ \frac{x_1}{6} + \frac{x_2}{9} \leq 1 \]
  
  - common sense requirements:
    \[ x_1 \geq 0, \ x_2 \geq 0 \]
PROBLEM STATEMENT

\[ \text{max } Z = 3x_1 + 5x_2 \]

\[ \text{s.t.} \]
\[ x_1 \leq 4 \]
\[ x_2 \leq 6 \]
\[ \frac{x_1}{6} + \frac{x_2}{9} \leq 1 \]
\[ x_1 \geq 0, x_2 \geq 0 \]
VISUALIZATION OF THE FEASIBLE REGION

\[ x_1 \geq 0, \quad x_1 \leq 4, \quad x_2 \geq 0 \]
VISUALIZATION OF THE
FEASIBLE REGION

\[ x_2 = 6 \]

\[ x_1 \geq 0, \; x_2 \leq 6, \; x_2 \geq 0 \]
VISUALIZATION OF THE
FEASIBLE REGION

\[ \frac{x_1}{6} + \frac{x_2}{9} \leq 1 \]

\[ x_1 \geq 0, \ x_2 \geq 0 \]
THE FEASIBLE REGION

\[ \frac{x_1}{6} + \frac{x_2}{9} = 1 \]

feasible region

\( (0,0) \)
\( (0,6) \)
\( (2,6) \)
\( (4,3) \)
\( (4,0) \)
\( x_1 = 4 \)
\( x_2 = 6 \)
CONTOURS OF CONSTANT $Z$

$\begin{align*}
(0,0) & \quad (2,2) \\
(0,6) & \quad (2,6) \\
(1.5,4.5) & \quad (4,3) \\
Z = 27 & \quad Z = 16 \\
\end{align*}$

$max Z = 3x_1 + 5x_2$

$\frac{x_1}{6} + \frac{x_2}{9} = 1$
CONTOURS OF CONSTANT $Z$

$Z = 10$

$Z = 15$

$Z = 20$

$Z = 25$

$Z = 36$

$(0,0)$

$(0,6)$

$(2,6)$

$(4,3)$

$(4,0)$

$(0,4)$

$x_1 = 4$

$x_2 = 6$
For this simple problem, we can graphically obtain the **optimal** solution

The **optimal** solution of this problem is:

\[
x^*_1 = 2 \quad \text{and} \quad x^*_2 = 6
\]

The objective value at the **optimal** solution is

\[
Z^* = 3x^*_1 + 5x^*_2 = 36
\]
LINEAR PROGRAMMING (LP)
PROBLEM DEFINITION

A linear programming problem is an optimization problem with a linear objective function and linear constraints.
EXAMPLE 3: ONE-POTATO, TWO-POTATO PROBLEM

- Mr. Spud manages the *Potatoes-R-Us Co.* which processes potatoes into packages of freedom fries (*F*), hash browns (*H*) and chips (*C*).

- Mr. Spud can buy potatoes from two sources; each source has distinct characteristics/limits.

- The problem is to determine the respective quantities Mr. Spud needs to buy from source 1 and from source 2 so as to maximize his profits.
EXAMPLE 3: ONE-POTATO, TWO-POTATO PROBLEM

The known data are summarized in the table:

<table>
<thead>
<tr>
<th>product</th>
<th>source 1 uses (%)</th>
<th>source 2 uses (%)</th>
<th>sales limit (tons)</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>20</td>
<td>30</td>
<td>1.8</td>
</tr>
<tr>
<td>H</td>
<td>20</td>
<td>10</td>
<td>1.2</td>
</tr>
<tr>
<td>C</td>
<td>30</td>
<td>30</td>
<td>2.4</td>
</tr>
<tr>
<td>profits ($/ton)</td>
<td>5</td>
<td>6</td>
<td>–</td>
</tr>
</tbody>
</table>

The following assumptions hold:

- 30% waste for each source
- Production may not exceed the sales limit
Decision variables:

\[ x_1 = \text{quantity purchased from source 1} \]
\[ x_2 = \text{quantity purchased from source 2} \]

Objective function:

\[ \text{max } Z = 5 \, x_1 + 6 \, x_2 \]

Constraints:

\[ 0.2 \, x_1 + 0.3 \, x_2 \leq 1.8 \quad (F) \]
\[ 0.2 \, x_1 + 0.1 \, x_2 \leq 1.2 \quad (H) \]
\[ x_1 \geq 0, \ x_2 \geq 0 \]
\[ 0.3 \, x_1 + 0.3 \, x_2 \leq 2.4 \quad (C) \]
FEASIBLE REGION DETERMINATION

freedom fries

hash browns

chips
THE FEASIBLE REGION
EXAMPLE 3: CONTOURS OF CONSTANT Z

max Z = 5x_1 + 6x_2

Z = 36
Z = 30
Z = 24
Z = 18
Z = 40.5

(4.5, 3)
The optimal solution of this problem is:

\[ x_1^* = 4.5 \quad \text{and} \quad x_2^* = 3 \]

The objective value at the optimal solution is:

\[ Z^* = 5x_1^* + 6x_2^* = 40.5 \]
IMPORTANT OBSERVATIONS

- Constant $Z$ lines are parallel and change monotonically along the direction normal to the contours of constant values of $Z$

- An *optimal* solution must be at one of the *corner points* of the feasible region: fortuitously, there are only a *finite* number of *corner points*

- If a particular *corner point* gives a better solution (in terms of the $Z$ value) than that at every other adjacent *corner point*, then, it is an *optimal* solution
CONCEPTUAL SOLUTION PROCEDURE

- Initialization step: start at a corner point
- Iteration step: move to an improved adjacent corner point and repeat this step as many times as needed
- Stopping rule: stop when the corner point solution is better than that at each adjacent corner point
- This conceptual procedure forms the basis of the simplex approach
EXAMPLE 3: THE SIMPLEX APPROACH SOLUTION

\[
\begin{align*}
\text{max } Z &= 5x_1 + 6x_2 \\
Z &= 36 \\
(0,6) &\quad 6 \\
Z &= 40 \\
(4.5,3) &\quad (6,0) \\
Z &= 30 \\
(0,0) &\quad 1 \\
(0,6) &\quad 6 \\
\end{align*}
\]
**EXAMPLE 3: THE SIMPLEX APPROACH SOLUTION**

<table>
<thead>
<tr>
<th>step</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$Z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>6</td>
<td>36</td>
</tr>
<tr>
<td>2</td>
<td>4.5</td>
<td>3</td>
<td>40.5</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>0</td>
<td>30</td>
</tr>
</tbody>
</table>
EXAMPLE 3: THE SIMPLEX APPROACH SOLUTION

\[ \max Z = 5x_1 + 6x_2 \]

Points:
- \((0, 6)\) with \(Z = 36\)
- \((4.5, 3)\) with \(Z = 40.5\)
- \((6, 0)\) with \(Z = 30\)
- \((0, 0)\) with \(Z = 0\)
EXAMPLE 3: THE SIMPLEX APPROACH SOLUTION

1. Start at \((0,0)\) with \(Z(0,0) = 0\)

2. (i) Move from \((0,0)\) to \((0,6)\), \(Z(0,6) = 36\)

   (ii) Move from \((0,6)\) to \((4.5,3)\); compute \(Z(4.5,3) = 40.5\)

3. Compare the objective at \((4.5,3)\) to values at \((6,0)\)
   and at \((0,6)\):

   \[
   Z(4.5,3) \geq Z(6,0)
   \]

   \[
   Z(4.5,3) \geq Z(0,6)
   \]

   therefore, \((4.5,3)\) is optimal
Key requirements of a programming problem:

- to make a decision and thus to define the decision variables
- to achieve the specified objective and thus to express mathematically the objective function
- to ensure that the decision satisfies all the constraints, which are mathematically formulated
REVIEW

- Key attributes of an LP
  - the objective function is \textit{linear}
  - the constraints are \textit{linear}

- Basic steps in formulating a programming problem
  - definition of decision variables
  - statement of the objective function
  - formulation of the constraints
REVIEW

- Words of caution: care is required with units and attention to not ignore the implicit constraints, such as nonnegativity, and the common sense requirements in an LP formulation

- Graphical solution approach for two-variable problems
  - feasible region determination
  - contours of constant $Z$
  - identification of the vertex with optimal $Z^*$
EXAMPLE 4: QUALITY CONTROL INSPECTION OF GOODS PRODUCED

- There are 8 grade 1 and 10 grade 2 inspectors available for QC inspection; at least 1,800 pieces must be inspected in each 8-hour day.

- Problem data are summarized below:

<table>
<thead>
<tr>
<th>grade level</th>
<th>speed (unit/hr)</th>
<th>accuracy (%)</th>
<th>wages ($/h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25</td>
<td>98</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td>95</td>
<td>3</td>
</tr>
</tbody>
</table>
EXAMPLE 4: INSPECTION OF GOODS PRODUCED

- Each error costs $2

- The problem is to determine the optimal assignment of inspectors, i.e., the number of inspectors of grade 1 and that of grade 2 to result in the least-cost QC inspection effort.
EXAMPLE 4: FORMULATION

- Definition of decision variables:
  \[ x_1 = \text{number of grade 1 inspectors assigned} \]
  \[ x_2 = \text{number of grade 2 inspectors assigned} \]

- Objective function
  - optimal assignment: minimum costs
  - costs = wages + errors
EXAMPLE 4: FORMULATION

• each grade 1 inspector costs:

\[ 4 + 2 (25)(0.02) = 5 \text{ \$/hr} \]

• each grade 2 inspector costs:

\[ 3 + 2 (15)(0.05) = 4.5 \text{ \$/hr} \]

• total daily inspection costs in \$\ are

\[ Z = 8 [5 x_1 + 4.5 x_2] = 40 x_1 + 36 x_2 \] (\$)
EXAMPLE 4: FORMULATION

Constraints:

- Job completion:
  \[ 8(25)x_1 + 8(15)x_2 \geq 1,800 \]
  \[ \iff 200x_1 + 120x_2 \geq 1,800 \]
  \[ \iff 5x_1 + 3x_2 \geq 45 \]

- Availability limit:
  \[ x_1 \leq 8 \]
  \[ x_2 \leq 10 \]

- Nonnegativity:
  \[ x_1 \geq 0, \quad x_2 \geq 0 \]
EXAMPLE 4: PROBLEM STATEMENT SUMMARY

Decision variables:

\[ x_1 = \text{number of grade 1 inspectors assigned} \]
\[ x_2 = \text{number of grade 2 inspectors assigned} \]

Objective function:

\[ \min Z = 40x_1 + 36x_2 \]

Constraints:

\[ 5x_1 + 3x_2 \geq 45 \]
\[ x_1 \leq 8 \]
\[ x_2 \leq 10 \]
\[ x_1 \geq 0, \ x_2 \geq 0 \]
MULTI – PERIOD SCHEDULING

- More than one period is involved
- The result of each period affects the initial conditions for the next period and therefore the solution
- We need to define variables to take into account the initial conditions in addition to the decision variables of the problem
EXAMPLE 5: HYDROELECTRIC POWER SYSTEM OPERATIONS

- We consider a single operator of a system consisting of two water reservoirs with a hydroelectric plant attached to each reservoir.
- We schedule the two power plant operations over a two-period horizon.
- We are interested in a plan to maximize the total revenues of the system operator.
EXAMPLE 5: HYDROELECTRIC POWER SYSTEM OPERATIONS

flows of water in the system
### EXAMPLE 5: $kAf$ RESERVOIR DATA

<table>
<thead>
<tr>
<th>parameter</th>
<th>reservoir A</th>
<th>reservoir B</th>
</tr>
</thead>
<tbody>
<tr>
<td>maximum capacity</td>
<td>2,000</td>
<td>1,500</td>
</tr>
<tr>
<td>predicted inflow in period 1</td>
<td>200</td>
<td>40</td>
</tr>
<tr>
<td>predicted inflow in period 2</td>
<td>130</td>
<td>15</td>
</tr>
<tr>
<td>minimum allowable level</td>
<td>1,200</td>
<td>800</td>
</tr>
<tr>
<td>level at start of period 1</td>
<td>1,900</td>
<td>850</td>
</tr>
</tbody>
</table>
EXAMPLE 5: SYSTEM CHARACTERISTICS

plant A  \[ 1 \text{ kA}\varnothing \rightarrow \text{plant A} \rightarrow 400 \text{ MWh} \]

plant B  \[ 1 \text{ kA}\varnothing \rightarrow \text{plant B} \rightarrow 200 \text{ MWh} \]

<table>
<thead>
<tr>
<th>reservoir</th>
<th>max kA\varnothing for generation per period</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>150</td>
</tr>
<tr>
<td>B</td>
<td>87.5</td>
</tr>
</tbody>
</table>
EXAMPLE 5: SYSTEM CHARACTERISTICS

- Two-tier price for the $MWh$ demand in each period

- Up to 50,000 $MWh$ can be sold @ 20 $/MWh$

- All additional $MWh$ are sold @ 14 $/MWh$

A non-linear objective function is depicted graphically, with a linear increase up to 50,000 $MWh$ at 20 $/MWh$, and a flat rate of 14 $/MWh$ thereafter.
### EXAMPLE 5: DECISION VARIABLES

<table>
<thead>
<tr>
<th>variable</th>
<th>quantity denoted</th>
<th>units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^i_H$</td>
<td>energy sold at 20 $/MWh</td>
<td>MWh</td>
</tr>
<tr>
<td>$x^i_L$</td>
<td>energy sold at 14 $/MWh</td>
<td>MWh</td>
</tr>
<tr>
<td>$w^i_A$</td>
<td>plant A water supply for generation</td>
<td>kAf</td>
</tr>
<tr>
<td>$w^i_B$</td>
<td>plant B water supply for generation</td>
<td>kAf</td>
</tr>
<tr>
<td>$s^i_A$</td>
<td>reservoir A spill</td>
<td>kAf</td>
</tr>
<tr>
<td>$s^i_B$</td>
<td>reservoir B spill</td>
<td>kAf</td>
</tr>
<tr>
<td>$r^i_A$</td>
<td>reservoir A end of period $i$ level</td>
<td>kAf</td>
</tr>
<tr>
<td>$r^i_B$</td>
<td>reservoir B end of period $i$ level</td>
<td>kAf</td>
</tr>
</tbody>
</table>

Superscript $i$ denotes period $i$, $i = 1, 2$
EXAMPLE 5: OBJECTIVE FUNCTION

maximize total revenues from sales

\[ \text{max } Z = 20(x_H^1 + x_H^2) + 14(x_L^1 + x_L^2) \]

4 of the 16 decision variables
2 for each period

units of $Z$ are in $\$
EXAMPLE 5: CONSTRAINTS

- Period 1 constraints
  - Energy conservation in a lossless system
    - Total generation: $400w_A^1 + 200w_B^1 \quad (MWh)$
    - Total sales: $x_H^1 + x_L^1 \quad (MWh)$
    - Losses are neglected and so:
      $$x_H^1 + x_L^1 = 400w_A^1 + 200w_B^1$$
  - Maximum available capacity limits
    $$w_A^1 \leq 150$$
    $$w_B^1 \leq 87.5$$
EXAMPLE 5 : CONSTRAINTS

- Reservoir conservation of flow relations

  **reservoir A:**

  \[
  w_A^1 + s_A^1 + r_A^1 = 1,900 + 200 = 2,100 (kA_f)
  \]

  - Res. level at e.o.p. 0
  - Predicted inflow

  **reservoir B:**

  \[
  w_B^1 + s_B^1 + r_B^1 = 850 + 40 + w_A^1 + s_A^1 (kA_f)
  \]

  - Res. level at e.o.p. 1
EXAMPLE 5: CONSTRAINTS

- limitations on reservoir variables

  - reservoir A:
    \[ 1,200 \leq r_{A}^{1} \leq 2,000 \]  \quad (kAf)

  - reservoir B:
    \[ 800 \leq r_{B}^{1} \leq 1,500 \]  \quad (kAf)

- sales constraint

  \[ x_{H}^{1} \leq 50,000 \]  \quad (kAf)
EXAMPLE 5: CONSTRAINTS

- Period 2 constraints

  - Energy conservation in a lossless system
    - Total generation: \( 400w_A^2 + 200w_B^2 \) (MWh)
    - Total sales: \( x_H^2 + x_L^2 \) (MWh)
    - Losses are neglected and so
      \[ x_H^2 + x_L^2 = 400w_A^2 + 200w_B^2 \]
  
  - Maximum available capacity limits
    \[ w_A^2 \leq 150 \]
    \[ w_B^2 \leq 87.5 \]
EXAMPLE 5: CONSTRAINTS

• reservoir conservation of flow relations

• reservoir $A$:

\[ w_A^2 + s_A^2 + r_A^2 = r_A^1 + 130 \quad (kAf) \]

- res. level at e.o.p. 2
- res. level at e.o.p. 1
- predicted inflow

• reservoir $B$:

\[ w_B^2 + s_B^2 + r_B^2 = r_B^1 + 15 + w_A^2 + s_A^2 \quad (kAf) \]
EXAMPLE 5: CONSTRAINTS

- limitations on reservoir variables

- reservoir A:

\[ 1,200 \leq r_A^2 \leq 2,000 \]

- reservoir B:

\[ 800 \leq r_B^2 \leq 1,500 \]

- sales constraint

\[ x_H^2 \leq 50,000 \]
EXAMPLE 5: PROBLEM STATEMENT

16 decision variables:

\[ x_H^i, x_L^i, w_A^i, w_B^i, s_A^i, s_B^i, r_A^i, r_B^i, \quad i = 1, 2 \]

Objective function:

\[ \max Z = 20(x_H^1 + x_H^2) + 14(x_L^1 + x_L^2) \]

Constraints:

- 20 constraints for the periods 1 and 2
- non-negativity constraints on all variables
EXAMPLE 6: DISHWASHER AND WASHING MACHINE PROBLEM

- The **Appliance Co.** manufactures dishwashers and washing machines.

- The sales targets for next four quarters are:

<table>
<thead>
<tr>
<th>product</th>
<th>variable</th>
<th>quarter t</th>
</tr>
</thead>
<tbody>
<tr>
<td>dishwasher</td>
<td>$D_t$</td>
<td>2,000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1,300</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3,000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1,000</td>
</tr>
<tr>
<td>washing</td>
<td>$W_t$</td>
<td>1,200</td>
</tr>
<tr>
<td>machine</td>
<td></td>
<td>1,500</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1,000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1,400</td>
</tr>
</tbody>
</table>
### EXAMPLE 6: QUARTERLY COST COMPONENTS

<table>
<thead>
<tr>
<th>cost component</th>
<th>parameter</th>
<th>quarter $t$ costs ($/unit)</th>
</tr>
</thead>
<tbody>
<tr>
<td>manufacturing ($/unit)</td>
<td>dishwasher</td>
<td>$c_t$</td>
</tr>
<tr>
<td></td>
<td>washing machine</td>
<td>$v_t$</td>
</tr>
<tr>
<td>storage ($/unit)</td>
<td>dishwasher</td>
<td>$j_t$</td>
</tr>
<tr>
<td></td>
<td>washing machine</td>
<td>$k_t$</td>
</tr>
<tr>
<td>hourly labor ($/hour)</td>
<td></td>
<td>$p_t$</td>
</tr>
</tbody>
</table>
EXAMPLE 6: CONSTRAINTS

- Each dishwasher requires 1.5 and each washing machine uses 2 hours of labor.
- The labor hours in each quarter cannot grow or decrease by more than 10%; there were 5,000 hours of labor in the quarter preceding the first quarter.
- At the start of the first quarter, there are 750 dishwasher washers and 50 washing machines in storage.
EXAMPLE 6 : THE PROBLEM

How to schedule the production in each of the four quarters so as to minimize the costs while meeting the sales targets?
**EXAMPLE 6: QUARTER $t$ DECISION VARIABLES**

<table>
<thead>
<tr>
<th>symbol</th>
<th>variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_t$</td>
<td>number of dishwashers produced</td>
</tr>
<tr>
<td>$w_t$</td>
<td>number of washing machines produced</td>
</tr>
<tr>
<td>$r_t$</td>
<td>final inventory of dishwashers</td>
</tr>
<tr>
<td>$s_t$</td>
<td>final inventory of washing machines</td>
</tr>
<tr>
<td>$h_t$</td>
<td>available labor hours during $Q_t$</td>
</tr>
</tbody>
</table>

$t = 1, 2, 3, 4$
EXAMPLE 6: OBJECTIVE FUNCTION

minimize the total costs for the four quarters

\[ \begin{align*}
\min Z &= c_1 d_1 + v_1 w_1 + j_1 r_1 + k_1 s_1 + p_1 h_1 & \text{quarter 1} \\
&+ c_2 d_2 + v_2 w_2 + j_2 r_2 + k_2 s_2 + p_2 h_2 & \text{quarter 2} \\
&+ c_3 d_3 + v_3 w_3 + j_3 r_3 + k_3 s_3 + p_3 h_3 & \text{quarter 3} \\
&+ c_4 d_4 + v_4 w_4 + j_4 r_4 + k_4 s_4 + p_4 h_4 & \text{quarter 4}
\end{align*} \]
EXAMPLE 6: CONSTRAINTS

- Quarterly flow balance relations:

\[ \begin{align*}
    r_{t-1} + d_t - r_t &= D_t \\
    s_{t-1} + w_t - s_t &= W_t
\end{align*} \]

\( t = 1, 2, 3, 4 \)
EXAMPLE 6: CONSTRAINTS

- Quarterly labor constraints

\[
\begin{align*}
1.5 d_t + 2 w_t - h_t & \leq 0 \\
0.9 h_{t-1} & \leq h_t & \leq 1.1 h_{t-1}
\end{align*}
\]

\[ t = 1, 2, 3, 4 \]

\[ h_0 = 5,000 \]
### EXAMPLE 6: PROBLEM STATEMENT

<table>
<thead>
<tr>
<th>$d_1$</th>
<th>$w_1$</th>
<th>$r_1$</th>
<th>$s_1$</th>
<th>$h_1$</th>
<th>$d_2$</th>
<th>$w_2$</th>
<th>$r_2$</th>
<th>$s_2$</th>
<th>$h_2$</th>
<th>$d_3$</th>
<th>$w_3$</th>
<th>$r_3$</th>
<th>$s_3$</th>
<th>$h_3$</th>
<th>$d_4$</th>
<th>$w_4$</th>
<th>$r_4$</th>
<th>$s_4$</th>
<th>$h_4$</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>-1</td>
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<td>1</td>
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</tbody>
</table>

Minimize: $1400 = 1000 - 1 
$3000 = 1500 - 1 
$1300 = 3000 - 1 
$1250 = 1300 - 1 
$1150 = 1250 - 1 
$5500 = 1150 - 1 
$4500 = 1000 - 1 

Constraints: 
$0 \leq 0 
4500 \geq 4500 
5500 \leq 5500 
1250 \leq 0 
1300 \geq 0 
1500 \leq 0 
3000 \leq 0 
1000 \geq 0 
1400 \leq 0 
1300 \geq 0 
1500 \leq 0 
$
LINEAR PROGRAMMING PROBLEM

\[ \text{max (min)} \quad Z = c_1 x_1 + \ldots + c_n x_n \]

\[ \text{s.t.} \]

\[ \begin{align*}
    a_{11} x_1 & + a_{12} x_2 & + \ldots & + a_{1n} x_n = b_1 \\
    a_{21} x_1 & + a_{22} x_2 & + \ldots & + a_{2n} x_n = b_2 \\
                  & \vdots \quad & & \vdots \\
    a_{m1} x_1 & + a_{m2} x_2 & + \ldots & + a_{mn} x_n = b_m \\
\end{align*} \]

\[ x_1 \geq 0, \; x_2 \geq 0, \ldots, \; x_n \geq 0 \]

\[ b_1 \geq 0, \; b_2 \geq 0, \ldots, \; b_m \geq 0 \]
STANDARD FORM OF $LP$ ($SFLP$)

$max \ (min) \ Z = c^T x$

$A x = b$

$x \geq 0$

coefficient matrix

decision vector

requirement vector

profits (costs) vector
CONVERSION OF LP INTO SFLP

An inequality may be converted into an equality by defining an additional nonnegative slack variable

- \( x_{slack} \geq 0 \)
- replace the given inequality \( \leq b \) by
  \[
  \text{inequality} + x_{slack} = b
  \]
- replace the given inequality \( \geq b \) by
  \[
  \text{inequality} - x_{slack} = b
  \]
CONVERSION OF LP INTO SFLP

- An unsigned variable $x_u$ is one whose sign is not specified.

- $x_u$ may be converted into two signed variables $x_+$ and $x_-$ with

$$
x_+ = \begin{cases} 
x_u & x_u \geq 0 \\
0 & x_u < 0
\end{cases}
$$

and

$$
x_- = \begin{cases} 
0 & x_u \geq 0 \\
-x_u & x_u < 0
\end{cases}
$$

so that $x_u$ is replaced by

$$
x_u = x_+ - x_-
$$
**SFLP CHARACTERISTICS**

- $x$ is feasible if and only if $x \geq 0$ and $Ax = b$

- $S = \{ x | Ax = b, x \geq 0 \}$ is the feasible region

- $S = LP$ is infeasible

- $x^*$ is optimal $c^T x^* \geq c^T x$, $x \in S$

- $x^*$ may be unique, or may have multiple values

- $x^*$ may be unbounded