$\Rightarrow O(m \log m)$.

update $C_{low}[\cdot,\cdot]$:

$$O\big((\deg(v_2)+\dots+\deg(v_\ell))\cdot\Delta\big)$$

$\xRightarrow{\text{total}} O(m\Delta \log m)$.

Rebuilding cost: $\tilde{O}\left(\frac{m\Delta}{q}\right)$ amort.

$\Rightarrow$ Overall amort update time

$$\tilde{O}\left(\frac{m\Delta}{q} + q + \frac{m}{\Delta} + \Delta^2\right)$$

Set $q = \Delta^2$: $\tilde{O}\left(\frac{m}{\Delta} + \Delta^2\right)$

Set $\Delta = m^{1/3}$: $\boxed{\tilde{O}(m^{2/3})}$ amort update ✓

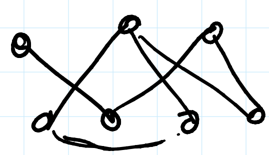
$\boxed{\tilde{O}(m^{1/3})}$ query

Why harder? Conditional Lower Bds ("Fine-Grained Complexity")

Problem (Triangle finding)

Given sparse graph $G=(V,E)$, $|E| = m$

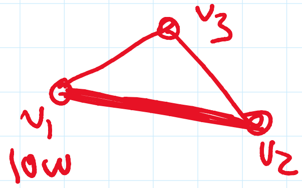
does G have a cycle $v_1 v_2 v_3$ of length 3?



(All-Edges version: $\forall v_1 v_2 \in E$, ? $\exists$ triangle thru $v_1 v_2$)

Alon-Yuster-Zwick's alg'm '94:

Case 1. one of v_1, v_2, v_3 has low deg $\le \Delta$.
say v_1



$\Rightarrow$ $O(m\Delta)$ time
by trying all m edges $v_1 v_2 \in E$
& neighbors v_3 of v_1

Case 2. all v_1, v_2, v_3 have high deg $> \Delta$

$\Rightarrow$ # high deg vertices $\le O\left(\frac{m}{\Delta}\right)$.

$\Rightarrow$ brute force $O\left(\left(\frac{m}{\Delta}\right)^3\right)$

total time $O\left(m\Delta + \left(\frac{m}{\Delta}\right)^3\right)$

$\Delta = \sqrt{m} \Rightarrow O(m^{3/2})$

can improve to $O\left(m\Delta + \left(\frac{m}{\Delta}\right)^{\omega}\right)$ ω = matrix mult. exponent < 2.38

$\Rightarrow O\left(m^{\frac{2\omega}{\omega+1}}\right) < O(m^{1.41})$

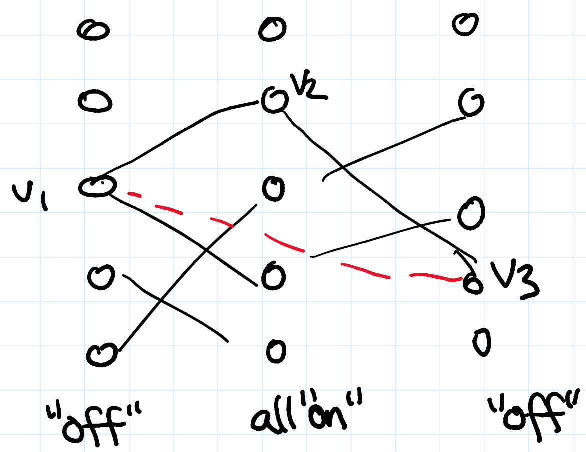
(if $\omega = 2$, $O(m^{4/3})$)

Conj: AE-Triangle finding requires $\Omega(m^{4/3-\varepsilon})$ time.

Thm If $\exists$ DS for dyn subgraph connectivity

Thm [C. '02] If $\exists$ DS for dyn subgraph connectivity can support $O(m^{1/3-\varepsilon})$ update & query time, then AE Triangle finding can be solved in $O(m^{4/3-\varepsilon})$ time.

Pf:



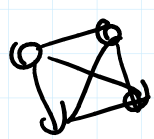
updates/queries $O(m)$

$\square$

[Patrascu '10: If $\exists$ alg'm for AE-Triangle finding, with $O(m^{4/3-\varepsilon})$ time,

then $\exists$ alg'm for 3SOM with $O(n^{2-\varepsilon'})$ time.]

Rmk - Jin, Xu '22: If $\exists$ dyn subgraph conn. with $O(m^{2/3-\varepsilon})$,



then 4-Clique in $O(n^{4-\varepsilon})$ time by "combinatorial algm".

Approx Distance Oracles

Problem Given undir. weighted graph $G=(V,E)$,

nonnegative

Problem Given undir. weighted graph $G=(V,E)$,
build static data structure s.t.
given $u,v \in V$, ~~find $d(u,v)$~~

shortest-path distance.

find value $\tilde{d}$ s.t.

$$d(u,v) \le \tilde{d} \le c\, d(u,v)$$

approx factor

Exact: APSP $\Rightarrow$ $\begin{cases} \text{preproc time} & O(n(n\log n + m)) \\ \text{space} & O(n^2) \\ \text{query time} & O(1) \end{cases}$

Approx: $(1+\varepsilon)$-approx APSP preproc $O(n^{2.38})$ (Zwick '99)

3-approx APSP $\tilde{O}(n^2)$ (Cohen-Zwick '01)

better data structures? Subquadratic space?

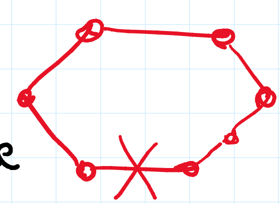
e.g. **Spanners**

$\exists$ subgraph of size $O(n^{1+1/k})$ that approximate all dists with factor $2k-1$

tight assuming Erdős girth conjecture ($\exists$ unwtd undir graph with $\Omega(n^{1+1/k})$ edges & girth $2k+1$)

$\Rightarrow$ good space bd

but not necessarily good query time



Thorup-Zwick '01:

preproc time $\boxed{O(k m n^{1/k})}$
space $\boxed{O(k n^{1+1/k})}$
query time $\boxed{O(k)}$
approx factor $\boxed{2k-1}$

$n^{\frac{1}{\log n}} = O(1)$

e.g. 3-approx, space $O(n^{3/2})$, query $O(1)$
5-approx, $n^{4/3}$
⋮
199-approx, $n^{1.01}$
$O(\log n)$-approx, space $O(n \log n)$, query $O(\log n)$

Thorup-Zwick Method

idea - random sampling

Fact Given a set A of n numbers, take a rand. subset $A' \subseteq A$ of $\frac{n}{b}$ numbers.

$$E\left[\left|\{x \in A - A' : x < \min(A')\}\right|\right] = O(b).$$