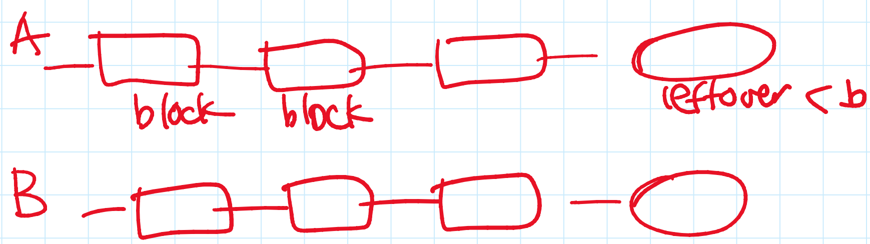
Better Method 3 : la Poutre '90

idea - recursion

let s = max set size

represent each set as a list of $\leq s/b$ blocks of size in $[b, 2b)$
plus a leftover set of size $< b$

store blocks using prev. method 1.2



union(A, B):
union(A's leftover, B's leftover)
use prev. method to union A's list of blocks & B's list of blocks

if leftover size $> b$
move leftover to new block z
for each $x \in z$, $\text{block}(x) = z$

find(x):
use prev. method to find set containing block(x)

Amort. Analysis: let $T(n,s)$ = total time of unions on n elements with max set size s

$$T(n,s) \leq T(n,b) + O\left(\frac{n}{b}\log\frac{s}{b} + n\right)$$

choose $b = \log s$ $\Rightarrow$ $T(n,s) \leq T(n, \log s) + O(n)$
$\Rightarrow O(n \log^* s)$

bootstrap $\Rightarrow$ $T(n,s) \leq T(n,b) + O\left(\frac{n}{b}\log^*\frac{s}{b} + n\right)$

choose $b = \log^* s \quad \Rightarrow \quad T(n,s) \le T(n, \log^* s) + O(n)$

$\Rightarrow \quad O(n \log^{**} s)$

$\vdots$

$$\boxed{\begin{aligned} T_k(n,s) &\le T_k(n,b) + T_{k-1}\left(\tfrac{n}{b}, \tfrac{s}{b}\right) + O(n) \\ Q_k(n,s) &\le Q_{k-1}\left(\tfrac{n}{b}, \tfrac{s}{b}\right) + O(1) \end{aligned}}$$

$\Rightarrow$ union $O(\log^{\overbrace{**\cdots*}^{k}} s)$ amortized

find $O(k)$

choose $k = \alpha(n) \quad \Rightarrow \quad \boxed{O(\alpha(n))}$

inverse Ackermann

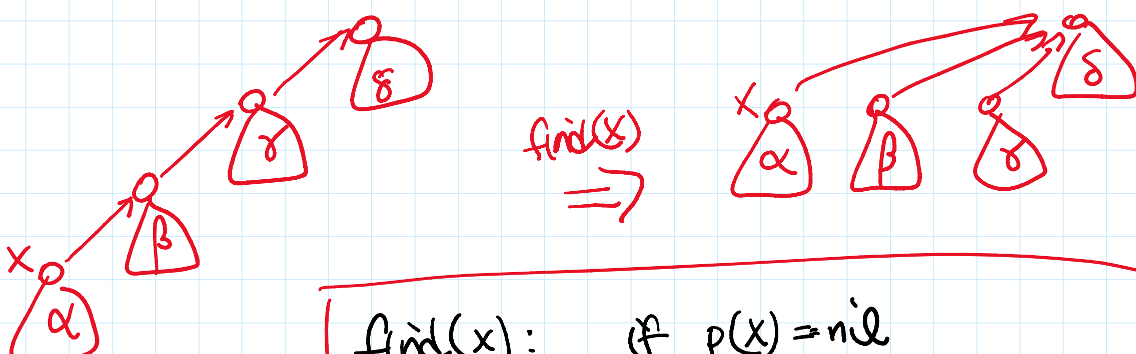
Better Method 4: Tarjan '75

back to tree method 2

use weighted union heuristic for union

+ path compression for find

Very simple to implement!!



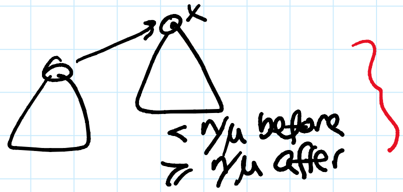
$$\boxed{\begin{aligned} &\text{find}(x): \quad \text{if } p(x) = \text{nil} \\ &\qquad\qquad\quad \text{return } x \\ &\qquad\qquad\quad \text{else return } p(x) = \text{find}(p(x)) \end{aligned}}$$

Obs 1 $\quad \text{size}(p(x)) \ge 2\,\text{size}(x)$

& $\text{size}(p(x))$ only increases

Obs 2 # nodes with $size(x) \geq \frac{n}{\mu}$ is $O(\mu)$.

Pf:



charge $\geq \frac{n}{\mu}$ nodes to x

(each node sends charge to ≤ 1 node) $\square$

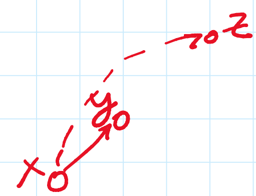
Amortized Analysis by Recurrence: (variant of Seidel-Sharir '05)

Consider interval $I = [\frac{n}{\mu}, s\frac{n}{\mu})$.

Say x is in I if $size(x) \in I$.

By Obs 2, # nodes in $I = O(\mu)$.

Let m = # finds whose answer is in I.



To bound $T(\mu, s, m)$ = # parent changes (i.e. changing $p(x)$ from y to z) with $x, y, z \in I$.

Divide I into $I_{low} = [\frac{n}{\mu}, b\frac{n}{\mu})$, $I_{high} = [b\frac{n}{\mu}, s\frac{n}{\mu})$

⋮