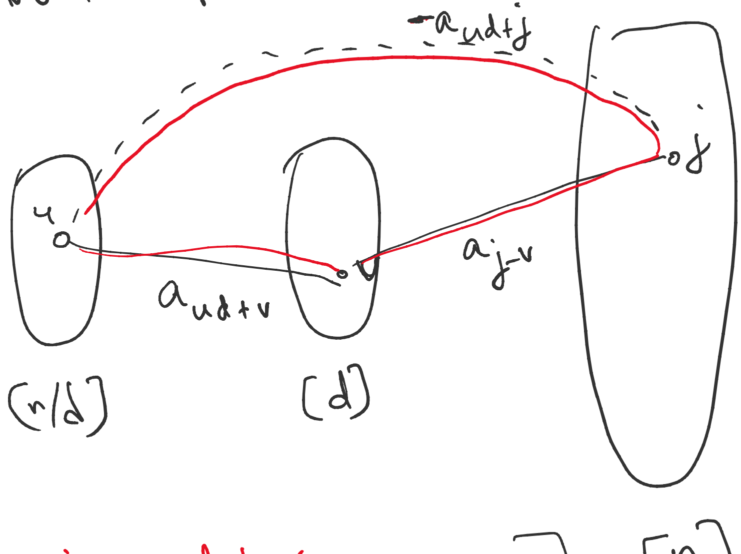
Assume Zero Triangle could be solved in $T(n) = O(n^{3-\delta})$ time.

To solve Convol3SUM for $a_1, \dots, a_n$:

Construct tripartite graph



$\exists$ zero-wt Δ

$\Rightarrow a_{ud+v} + a_{j-v} = a_{ud+j}$

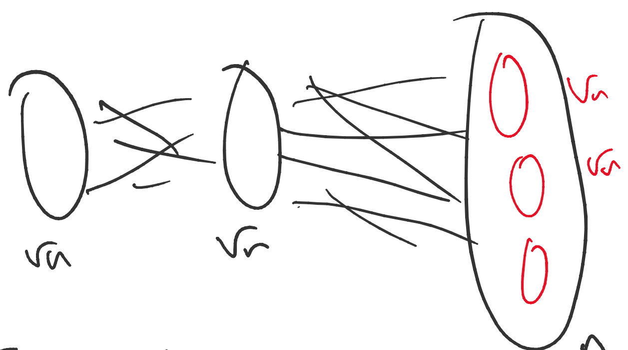
$a_i + a_j = a_{i+j}$

$i = ud+v$, $j \to j-v$, $i+j \to ud+j$

Write $i = ud+v$

$u \in [n/d]$, $v \in [d]$

Set $d = \sqrt{n}$



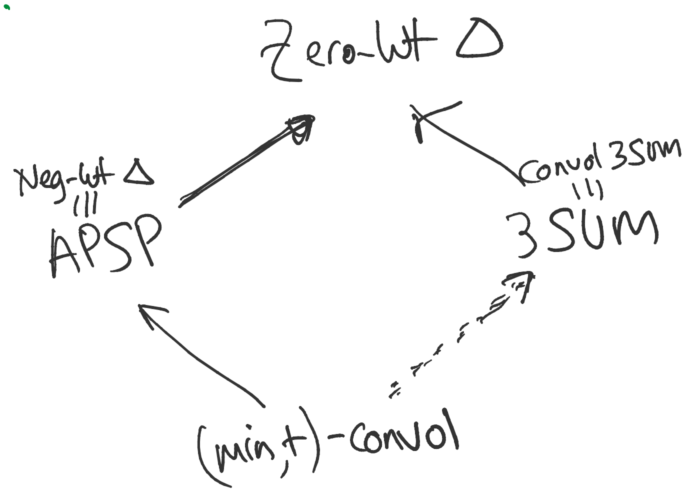
break into $\sqrt{n}$ graphs with $O(\sqrt{n})$ vertices each

$$\Rightarrow \text{time } O(\sqrt{n} \cdot T(\sqrt{n}))$$
$$= O(\sqrt{n} \cdot (\sqrt{n})^{3-\delta})$$
$$= O(n^{2-\delta/2}). \quad \square$$

Rmk:

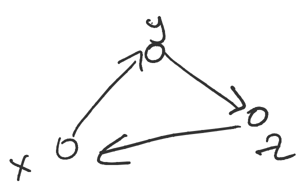
Zero-Wt Δ

Rmk.



Rmk: one could reduce Zero-wt Triangle back to 3SUM
but bds are weaker

(given tripartite graph G,
define 3 sets of numbers



$$A = \{ w(xy) + xM - yM^2 \mid xy \in E,\ x \in X, y \in Y \}$$

$$B = \{ w(yz) + yM^2 - zM^3 \mid yz \in E,\ y \in Y, z \in Z \}$$

$$C = \{ w(zx) + zM^3 - xM \mid xz \in E,\ x \in X, z \in Z \}$$

if 3SUM could be solved in $T(n)$ time,
$= O(n^{3/2-\delta})$

Zero-wt Δ could be solved
in $O(T(n^2))$ time
$= O(n^{3-\delta'})$.

and thus APSP

Triangle Listing Problem

Given unweighted graph G with m edges,
report all K triangles.

Recall: for triangle detection (K=1)
Alon, Yuster, Zwick '97 ...

Recall: for [illegible]

Alon, Yuster, Zwick '97

time $O\left(m^{\frac{2\omega}{\omega+1}}\right) = O(m^{1.41})$

(if $\omega = 2$, $O(m^{4/3})$)

for triangle listing,

Bjorklund et al. '14 Same bound for all $K \le m$.

Thm (Patrascu '10)

Assuming (int) 3SUM conj,

no $O(m^{4/3-\delta})$ alg'm for triangle listing with $K = O(m)$.

Intermediate Problem: Set Intersection Queries

Given N sets $S_1, \ldots, S_N$,

build data structure to answer queries of this type:

given (i,j), report $S_i \cap S_j$

in offline case: all queries are given in advance

Reduction: Convol3SUM $\rightarrow$ Offline-Set-Intersect Queries

To solve Convol3SUM for $a_1, \ldots, a_n$:

use hash fn $h(x) = x \bmod p$ for rand $p \in (R/2, R]$

let bucket $B_\ell = \{ j : h(a_j) = \ell \}$ $(\ell \in [R])$

Call bucket B_ℓ good if its size is $\tilde{O}\left(\frac{n}{R}\right)$

(& Prob[ans in 3 good buckets] $= \Omega(1)$)

(& $\text{Prob}[\text{ans in 3 good buckets}] = \Omega(1)$)

$a_i + a_j = a_{i+j}$

$h(a_i) + h(a_j) = h(a_{i+j})$ or $h(a_{i+j}) - p$

1. for $i = 1$ to n do
2. for $\ell \in [R]$ do
3. find all j s.t. $\underbrace{h(a_j) = \ell}_{j \in B_\ell}$ and $\underbrace{h(a_{i+j}) = \ell + h(a_i)}_{i+j \in B_{\ell + h(a_i)}}$ or $\ell + h(a_i) - p$
4. test if $a_i + a_j = a_{i+j}$ for each such j (exit if yes)

line 3 reduces to intersecting B_ℓ and $B_{\ell + h(a_i)} - i$

$S + x = \{ s + x \mid s \in S \}$

$\#$ sets $= O(Rn)$ too big.

idea - write $i = ud + v$, $u \in [n/d]$, $v \in [d]$

reduces to intersecting $\underbrace{B_\ell + v}$ and $\underbrace{B_{\ell + h(a_i)} - ud}$

$\#$ diff sets $O(Rd)$ $\qquad$ $\#$ diff sets $O(R\frac{n}{d})$

Choose $d = \sqrt{n}$

$\Rightarrow$ $N = \#$ sets $= O(R\sqrt{n}) = O(n)$

total set size $= \tilde{O}(R\sqrt{n} \cdot \frac{n}{R}) = \tilde{O}(n^{3/2})$

$\#$ queries $= O(nR) = O(n^{3/2})$

total output size of the queries

total output size of the queries

$= \#$ false positives

$= \#\,(i,j)$ with $h(a_i)+h(a_j) = h(a_{i+j})$ or $h(a_{i+j}) - p$ but $a_i + a_j \neq a_{i+j}$.

Set $R = \sqrt{n}$

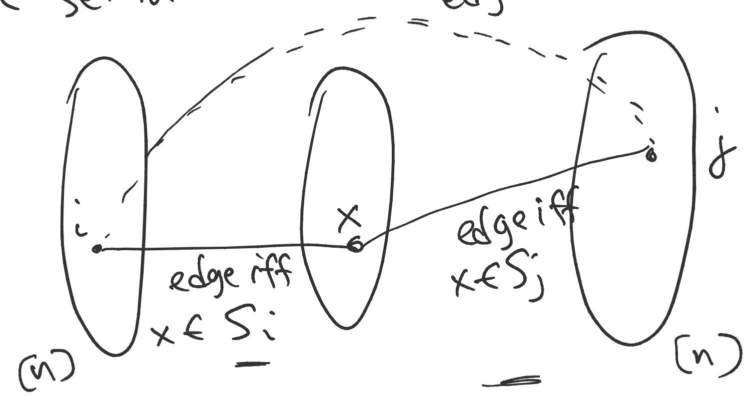
$= \tilde{O}\left(\frac{n^2}{R}\right)$ expected.

$= \tilde{O}(n^{3/2})$

Reduction: Offline Set-Intersection Q $\rightarrow$ Triangle Listing

Suppose listing $O(m)$ triangles in graph with m edges takes $T(m) = O(m^{4/3-\delta})$ time.

To solve Set-Intersect:



$\#$ edges $= \tilde{O}($total set size $+ \#$ queries$)$

$= \tilde{O}(n^{3/2})$

$\#$ triangles $=$ total output size of queries

$= \tilde{O}(n^{3/2})$

$\Rightarrow$ time $T(\tilde{O}(n^{3/2})) = \tilde{O}\left((n^{3/2})^{\frac{4}{3}-\delta}\right)$

$= \tilde{O}(n^{2-\frac{3}{2}\delta})$

$$= \tilde{O}\left(n^{2.\text{[illegible]}20}\right). \quad \square$$

Rmk: Vassilevska W. - Xu '20 showed APSP $\rightarrow$ Triangle Listing . . .