October 28, 2020 10:42 AM

Conditional Lower Bds Based on 3SUM

3SUM Problem Given set S of n numbers,
decide if $\exists a,b,c \in S$ s.t. ~~$a+b+c=0$~~ $a+b=c$

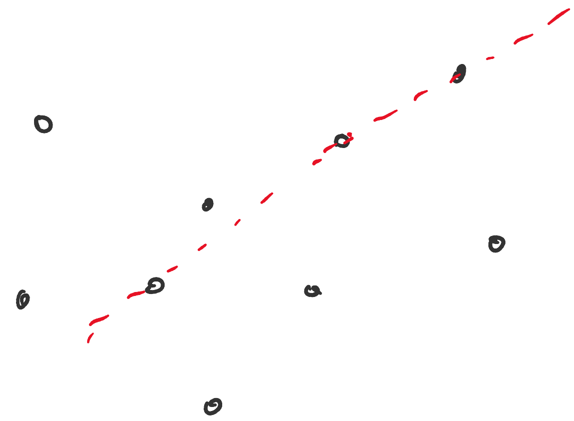
(3-set version: given A,B,C, $\exists a \in A, b \in B, c \in C$ s.t.
$a+b+c=0$)

Conjecture no $O(n^{2-\delta})$-time algm for 3SUM
(for reals or for ints)
(or more strongly, for ints in $[n^2]$)

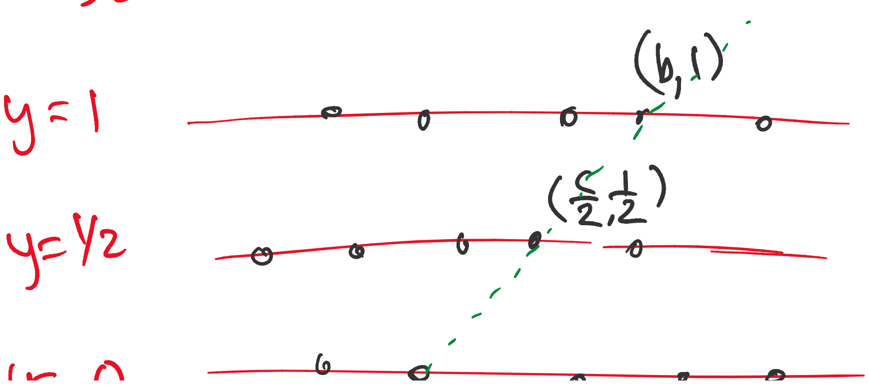
History: Gajentaan-Overmars '93 in computational geometry

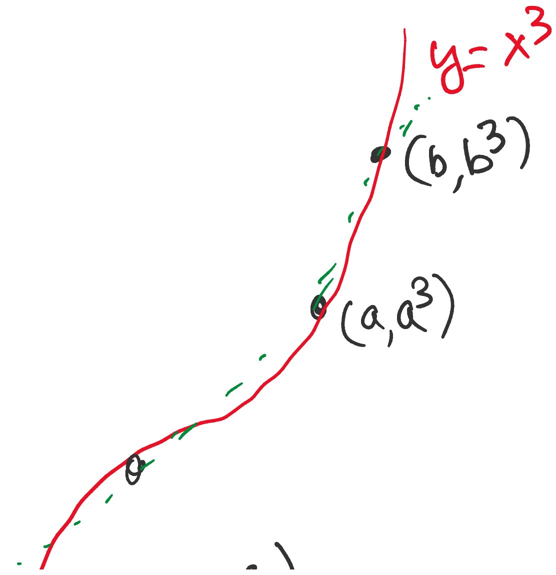
Exs of Geometric Problems

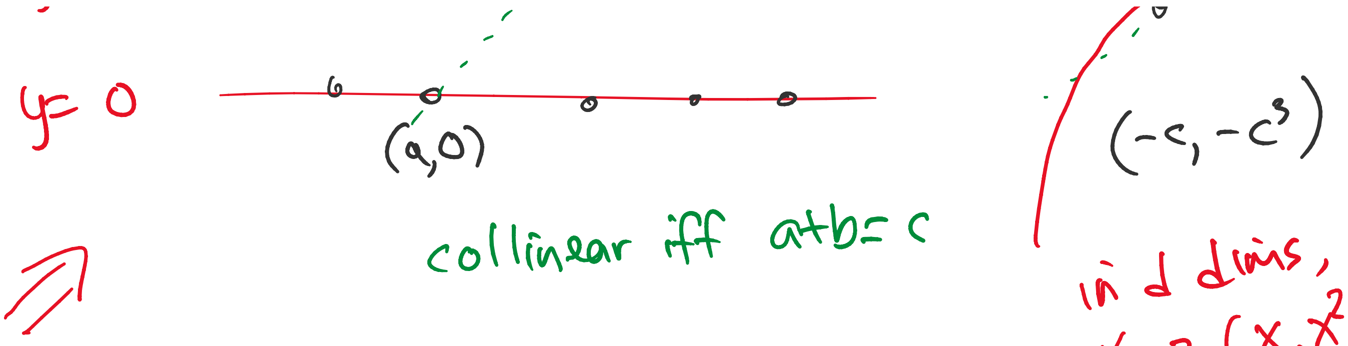
3-Collinear-Pts: (affine degeneracy testing) given set S of n pts in 2D,
decide if $\exists$ 3 pts of S lying on a common line



3SUM $\to$ 3-Collinear-Pts



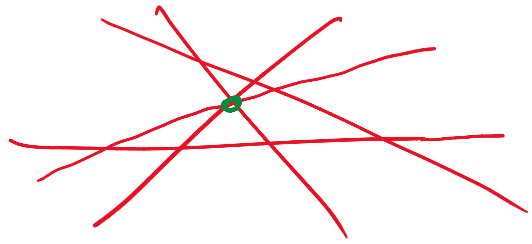




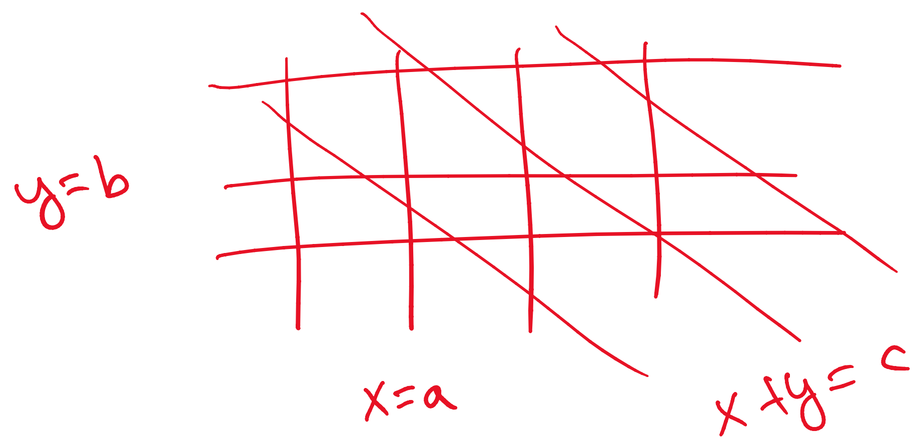
collinear iff $a+b=c$

in d dims,
$x \to (x, x^2, \ldots, x^{d-1}, x^{d+1})$
(Jeff: "weird moment curve")

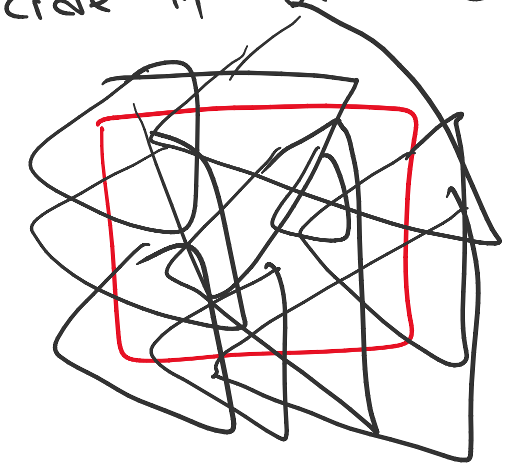
3-Concurrent Lines: Given n lines in 2D,
decide $\exists$ 3 lines that intersect at a common pt



3SUM $\to$ 3-Concurr-Lines

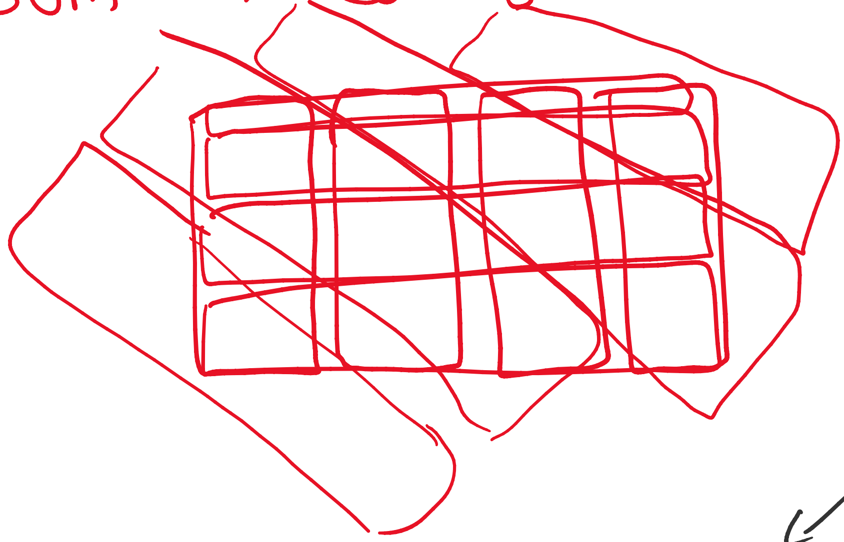


Coverage: Given n objects in 2D, (triangles, ...)
decide if union covers $[0,1]^2$



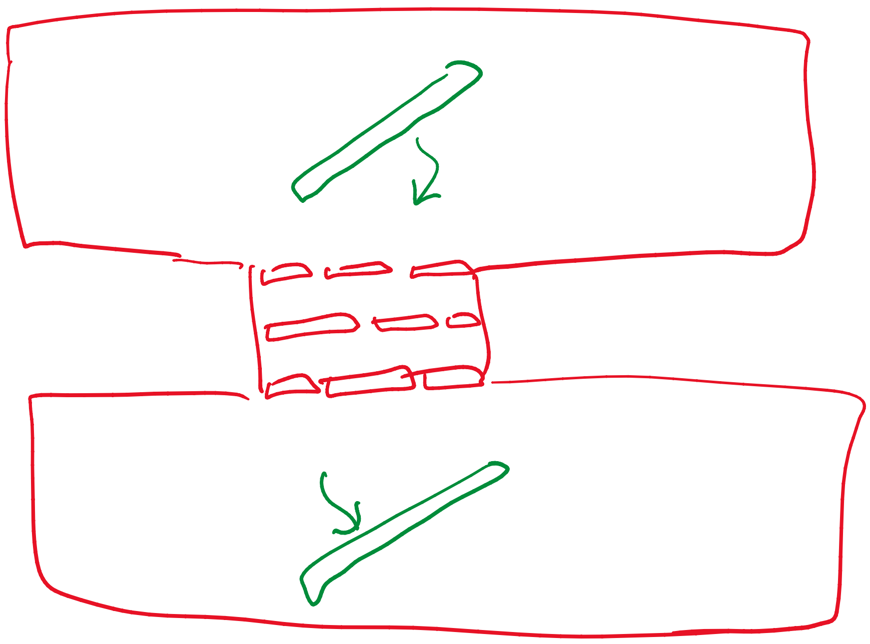
3SUM $\to$ Coverage

3SUM → Coverage



polygon in 2D

Motion Planning: given n obstacles & robot, decide if robot can be moved from one position to another



Etc.

Thm (Patrascu '10) Assuming (int) 3SUM conj, no $O(n^{3-\delta})$ alg'm for Zero-Wt Triangle for weighted graphs

Rmk: we proved this before assuming APSP conj.

Convolution-3SUM Problem Given $a_1, \ldots, a_n$,

[illegible]

Convolution-3SUM Problem   Given ...

decide if $\exists\, i,k$ s.t. $a_i + a_{k-i} = a_k$ ←

(i.e. $\exists\, i,j$ s.t. $a_i + a_j = a_{i+j}$)

Obviously, Convol3SUM → 3SUM.

(map $a_i \to (i, a_i)$)

$\to iM + a_i$ for large M)

Reduction: 3SUM → Convol-3SUM (for ints)

(Patrascu'10 / Kopelowitz-Pettie-Porat '16 / C.-He '20)

idea - hashing, by a "almost" linear fn

$(h(a+b) = h(a) + h(b))$

e.g. pick rand. prime $p \in [R/2, R]$

let $h(x) = x \bmod p$.

15	5
12	2
27	7
15	5
16	6
31	11

Prop (i) $h(a+b) = h(a) + h(b)$ or $h(a) + h(b) - p$

(ii) for fixed $a, a' \in [U]$ with $a \neq a'$,

$$\Pr[\, h(a) = h(a') \,] \leq \tilde{O}\left(\frac{1}{R}\right).$$

Pf of (ii):

$$= \Pr[\, a \equiv a' \bmod p \,]$$

$$= \Pr[\, p \text{ is prime divisor of } a - a' \,]$$

$$= \frac{\#\text{ prime divisors of } a - a'}{\#\text{ primes}}$$

$$= \Theta\left(\frac{\log U}{}\right) = \tilde{\Theta}\left(\frac{1}{R}\right)$$

$$= \frac{\Theta(\log U)}{\Theta(R/\log R)} = \tilde{\Theta}\left(\frac{1}{R}\right). \quad \square$$

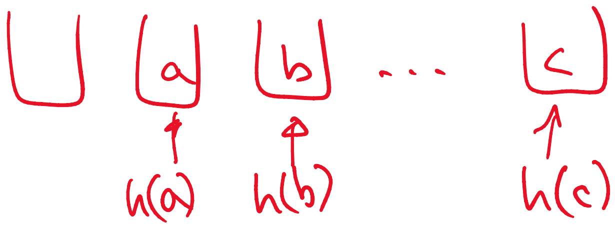
Cor for each fixed ℓ, set S of n numbers,
the "bucket" $B_\ell = \{a \in S : h(a) = \ell\}$
has expected size $\tilde{O}\left(\frac{n}{R}\right)$.

To solve 3SUM for set S of n numbers:

Choose $R = n$.

Call bucket B_ℓ "good" if its size is $\tilde{O}(1)$.

ans is in 3 good bucket is $\Omega(1)$ prob.



for each nonempty bucket B_ℓ do
pick $x \in B_\ell$ at rand.
& set x's index to ℓ
(i.e. $a_\ell = x$).

Solve Convol3SUM on $a_1, \ldots, a_n$.

ans found in $\Omega(1)$ prob.

repeat $\tilde{O}(\log n)$ times. $\square$

Rmk: can lower the extra log (C.-He '20)
or derandomize ...

Reduction: Convol3SUM → Zero-Wt Triangle

idea - Similar to $(\min,+)$-Convol → $(\min,+)$-MM idea – similar to

$$
\begin{array}{cc}
 & d \\
\frac{n}{d} & \begin{bmatrix} a_0 & \cdots & a_{d-1} \\ a_d & \cdots & a_{2d-1} \\ & \vdots & \end{bmatrix}
\end{array}
\quad
\begin{array}{cc}
 & n \\
d & \begin{bmatrix} b_0 & b_1 & b_2 & & b_{d-1} & b_d & & b_{n-1} \\ & b_0 & b_1 & & \vdots & \vdots & \cdots & \\ & & b_0 & & \vdots & \vdots & & \\ & & & & b_0 & b_1 & & b_{n-d} \end{bmatrix}
\end{array}
=
$$

$d = \sqrt{n}$

$$M^*(\sqrt{n}, \sqrt{n}, n)$$

$$= O\big(\sqrt{n}\ \underbrace{M^*(\sqrt{n})}\big)$$

$$\sqrt{n}\,(\sqrt{n})^{3-\delta} = n^{2-\delta/2}$$