Conditional Lower Bds Based on 3SUM

3SUM Problem
Given set S of n numbers, decide if \( \exists a, b, c \in S \) st. \( a + b + c = 0 \)

(3-set version: given \( A, B, C \), \( \exists a \in A, b \in B, c \in C \) st. \( a + b + c = 0 \))

Conjecture
no \( O(n^{2-\delta}) \)-time algm for 3SUM
(for reals or for ints)
(or more strongly, for ints in \( \mathbb{Z}^2 \))

History: Gajentaan-Overmars '93 in computational geometry

Exs of Geometric Problems

3-Collinear-Pts: given set \( S \) of \( n \) pts in \( \mathbb{Z}D \), decide if \( \exists 3 \) pts of \( S \) lying on a common line

\( 3 \text{SUM} \rightarrow 3 \text{-Collinear-Pts} \)

\( y = 1 \)
\( y = \sqrt{2} \)
\( y = x^3 \)
collinear iff $a + b = c$

3-Concurrent Lines: Given $n$ lines in 2D, decide if $3$ lines that intersect at a common pt

3-SUM $\rightarrow$ 3-Concurrent Lines

Coverage: Given $n$ objects in 2D, decide if union covers $[0,1]^2$
Motion Planning: given n obstacles & robot, decide if robot can be moved from one position to another

Etc.

Thm (Patrascu'10) Assuming (int) 3SUM conj, no $O(n^{3-\epsilon})$ alg'n for Zero-WT Triangle for weighted graphs

Rmk: we proved this before assuming APSP conj.

Convolution-3SUM Problem

Given $a_1, \ldots, a_n$, $a_1 \ast a_2 \ast \cdots \ast a_n = a_n$
Convolution-3Sum problem

Given \( \mathbf{a} \), decide if \( \exists i, k \) s.t.
\[
\mathbf{a} + \mathbf{a}_k = \mathbf{a} \tag{1}
\]
(i.e. \( \exists i, j \) s.t. \( a_i + a_j = a_{i+j} \))

Obviously, \( \text{Convol-3Sum} \rightarrow 3 \text{Sum} \).
( map \( a_i \rightarrow (i, a_i) \))
\[
\Rightarrow iM + a_i \quad \text{for large } M
\]

Reduction: \( 3 \text{Sum} \rightarrow \text{Convol-3Sum} \) (for wts)
(Patrascu '10 / Kopelowitz-Pettie-Porat '16 / C.-He '20)

Idea - hashing, by a linear fn
\[
h(a+b) = h(a) + h(b)
\]

E.g.
pick rand. prime \( p \in [R/2, R] \)
let \( h(x) = x \mod p \).

Prop (i) \[
h(a+b) = h(a) + h(b)
\]
\[
\text{or } h(a) + h(b) = p
\]

(ii) for fixed \( a, a' \in [0] \) with \( a \neq a' \),
\[
\Pr \left[ h(a) = h(a') \right] \leq \tilde{O}(\frac{1}{R})
\]

Pf of (ii):
\[
\begin{align*}
\Pr & \left[ a \equiv a' \mod p \right] \\
& = \Pr \left[ p \text{ is prime divisor of } a - a' \right] \\
& = \# \text{ prime divisors of } a - a' \\
& \quad \frac{1}{\# \text{ primes}} \\
& = \Theta \left( \log U \right) = \tilde{O}(\frac{1}{R})
\end{align*}
\]
\[= \frac{\Theta((\log U)}{\Theta(R/\log R)} = \tilde{\Theta}\left(\frac{1}{R}\right). \]

**Cor** for each fixed \( l \), set \( S \) of \( n \) numbers.
the "bucket" \( B_{l} = \{ a \in S : h(a) = l \} \)
has expected size \( \tilde{\Theta}\left(\frac{n}{R}\right) \).

To solve 3SUM for set \( S \) of \( n \) numbers:

Choose \( R = n \).
Call bucket \( B_{l} \) "good" if its size is \( \tilde{\Theta}(1) \).
\(< \) ans is in 3 good bucket is \( \Omega(1) \) prob.

\( \bigcup \{a, b, \ldots, c\} \)
for each nonempty bucket \( B_{l} \), do
pick \( x \in B_{l} \) at rand.
& set \( x \)'s index to \( R \)
(i.e. \( a_{R} = x \)).

Solve \textit{Convolsum} on \( a_{1}, \ldots, a_{n} \).
\( \uparrow \)
ans found in \( \tilde{\Theta}(1) \) prob.

Repeat \( \tilde{\Theta}(\log n) \) times.

**Remk:** can lower the extra log \( \tilde{\Theta}(\log n) \)
or derandomize ...

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\textbf{Reduction!} \textit{Convolsum} \( \rightarrow \) \textit{zero-\text{lat} Triangle}

idea - Similar to \((\min,+)\)-Convolution \( \rightarrow \)(\(\min,+)\)-MM
\[ d = \sqrt{n} \]

\[ M^*(V_n, \sqrt{n}, n) \]

\[ = O \left( \frac{M^*(V_n)}{V_n (V_n)^{3-\delta}} \right) \]

\[ V_n (V_n)^{3-\delta} = n^{2 \cdot \frac{\delta}{2}} \]