October 7, 2020 10:45 AM

Conditional Lower Bds via APSP

APSP

$\downarrow$

(min,+)-MM

$\downarrow$

All-Pairs Negative-Wt Triangle

$\downarrow$

Negative-Wt Triangle $\rightarrow$ Graph Radius $\left(\min_s \max_t d(s,t)\right)$

$\downarrow$

Zero-Wt Triangle

Problem: Zero-Wt Triangle (Exact Triangle, or Love Triangle)

test if $\exists\, x,y,z$ s.t. $w(xy) + w(yz) + w(xz) = 0$

Reduction: Negative Triangle $\rightarrow$ Zero Triangle (Vassilevska W., Williams '13)

assumes int. in $[U]$

idea - turn inequality to equality

Lemma

$$x+y+z > 0 \qquad (x,y,z \in [U])$$

$$\Longleftrightarrow \exists i, \left\lfloor \frac{x}{2^i} \right\rfloor + \left\lfloor \frac{y}{2^i} \right\rfloor + \left\lfloor \frac{z}{2^i} \right\rfloor = 1 \text{ or } 2 \text{ or } 3 \ldots \text{or } 7$$

Pf: $(\Leftarrow)$ obvious $\left(\frac{x}{2^i} + \frac{y}{2^i} + \frac{z}{2^i} \geq \left\lfloor \frac{x}{2^i} \right\rfloor + \left\lfloor \frac{y}{2^i} \right\rfloor + \left\lfloor \frac{z}{2^i} \right\rfloor > 0 \right)$

$(\Rightarrow)$ Say $2^{i+2} \leq x+y+z < 2^{i+3}$ $(i \geq 0)$

Then

$$1 \le \frac{x+y+z}{2^i} - 3 < \left\lfloor \frac{x}{2^i} \right\rfloor + \left\lfloor \frac{y}{2^i} \right\rfloor + \left\lfloor \frac{z}{2^i} \right\rfloor \le \frac{x+y+z}{2^i} < 8$$

□

Suppose Zero Triangle could be solved in $T(n)$ time

Then Neg Triangle in $O(T(n)\log U)$ time.

□

Consequence: APSP reduces to Zero Triangle.

(later, 3SUM reduces to Zero Triangle) ...

Reduction: Neg Triangle → Graph Radius

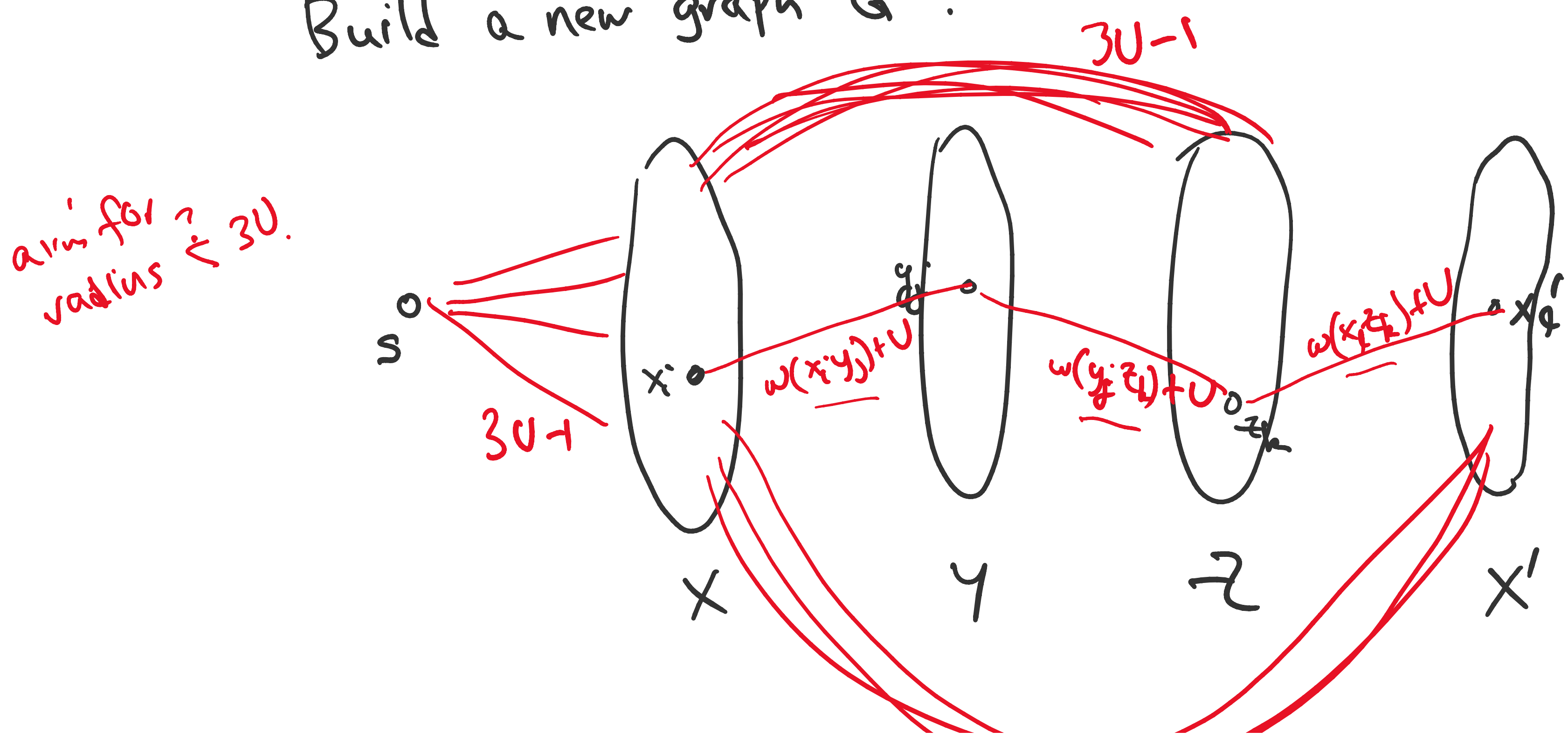
(Abboud, Grandoni, Vassilevska W. '15)

Suppose graph radius could be computed in $T(n)$ time.

To solve Neg Triangle for a given tripartite graph $G = (X \cup Y \cup Z, E)$:

Suppose edge wts are in $\{-U+1, \dots, U-1\}$.

Build a new graph $\hat{G}$:



$3U-1$ unless $i=\ell$

Compute radius of $\hat{G}$ in $T(O(n))$ time

Claim $X \times Y \times Z$ has neg triangle in G
$\iff$ $\hat{G}$ has radius $< 3U$.

Pf: $(\Rightarrow)$ Say x_i, y_j, z_k is neg triangle.
Pick x_i as center.
dist from x_i to any vertex $< 3U$.

$(\Leftarrow)$ Suppose radius $< 3U$.
Center must be in X.
x_i
dist from x_i to $x_i' < 3U$
$\Rightarrow \exists$ neg triangle in G. □

Rmk: many other probs: eccentricity, betweenness centrality, ...

Open: APSP $\to$ diam?

Conditional LBs Based on $(\min,+)$-Convolution

Problem Given $a_0, .., a_{n-1}, b_0, .., b_{n-1}$ (real or inf).
Compute $c_k = \min_i (a_i + b_{k-i})$ $k=0,..,2n-2$

Conjecture No $O(n^{2-\delta})$ alg'm for $(\min,+)$-convol

Conjecture No $O(n^{2-\delta})$ alg'm for $(\min,+)$-convol

this is a stronger than APSP conjecture

Reduction: $\underset{\text{(in subquad.)}}{(\min,+)\text{-Convol}} \longrightarrow \underset{\text{in subcubic}}{(\min,+)\text{-MM}}$

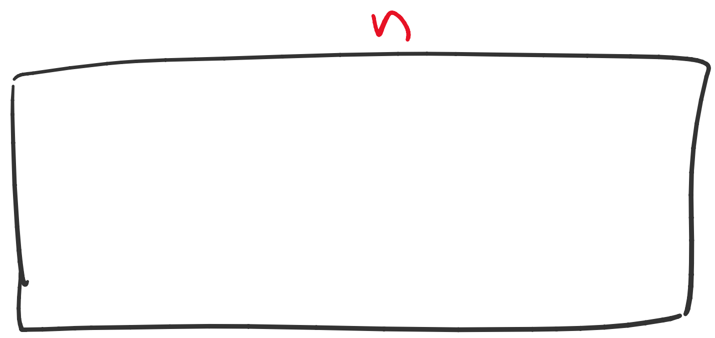
(BCDEHILPT '14)

Suppose $(\min,+)$ MM can be solved in $M^*(n) = O(n^{3-\delta})$ time.

To solve $(\min,+)$-Convol:

$$
\frac{n}{d}\left\{\begin{bmatrix} a_0 & \cdots & a_{d-1} \\ a_d & \cdots & a_{2d-1} \\ & \vdots & \\ a_{n-d} & & a_{n-1} \end{bmatrix}\right. \underbrace{}_{d} * \; d \begin{bmatrix} b_0 & b_1 & b_2 & & b_{d-1} & b_d & b_{d+1} & \cdots & b_{n-1} \\ 0 & b_0 & b_1 & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & b_0 & & & & & & \\ 0 & 0 & \vdots & & b_0 & b_1 & b_2 & & b_{n-d} \end{bmatrix}_{n}
$$

$$= \frac{n}{d}$$

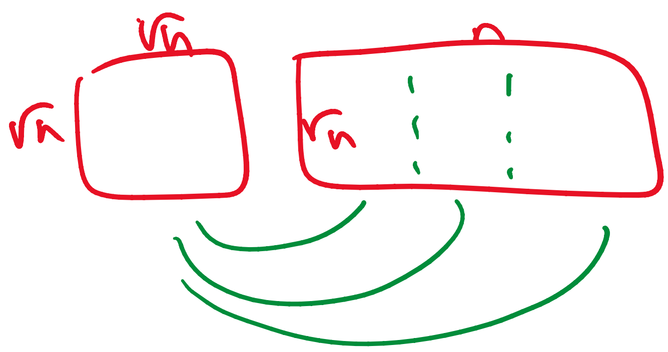


each c_k can be computed from $\underline{\frac{n}{d}}$ output entries

$\Rightarrow$ time $O\left(\cancel{n \cdot \frac{n}{d}} + M^*\left(\frac{n}{d}, d, n\right)\right)$

time for rect $(\min,+)$-MM.

Set $d = \sqrt{n}$



$$= O\left(M^*(\sqrt{n}, \sqrt{n}, n)\right)$$
$$= O\left(\sqrt{n}\ \underbrace{M^*(\sqrt{n})}\right)$$
$$= O\left(\sqrt{n}\ (\sqrt{n})^{3-\delta}\right)$$
$$= O\left(n^{2-\delta/2}\right). \qquad \square$$

0/1 Knapsack Problem

Given n items (w_i, p_i), $i = 1, \ldots, n$,

weight, profit (all positive ints)

and t ← capacity

find subset $I \subseteq \{1, \ldots, n\}$ to maximize $\sum_{i \in I} p_i$ s.t. $\sum_{i \in I} w_i \le t$.

Unbounded Knapsack: may use item multiple times (I is a multiset)

Standard DP: $O(nt)$ time

Subset sum is a special case of 0/1 Knapsack ($p_i = w_i$).

Could there be an $\tilde{O}(t)$ alg'm?

Thm (Cygan et al. '17 / Künnemann et al. '17)

... $O(n^{2-\delta})$ alg'm for some $\delta > 0$

Thm

$(\min,+)$-Convol has $O(n^{2-\delta})$ alg'm for some $\delta>0$

$\iff$ 0/1 Knapsack has $O(t^{2-\delta'})$ alg'm for some $\delta'>0$.

$\iff$ Unbdd Knapsack has $O(t^{2-\delta''})$ alg'm for some $\delta''>0$.

$\vdots$

Pf: $(\Rightarrow)$ reduce Unbdd Knapsack $\longrightarrow$ $(\max,+)$-Convol

by repeated squaring

$$f^{(\ell)}[j] = \max_{j'} \Big(\underbrace{f^{(\ell/2)}[j'] + f^{(\ell/2)}[j-j']}_{(\max,+)\text{-Convol on input of size } t} \Big)$$

max profit for capacity j with $\leq \ell$ items ($j=0,\dots,t$)

(max,+)-Convol !! on input of size t

reduce 0/1 Knapsack to $(\max,+)$-Convol by modifying Bringman's subset sum alg.

$(\Leftarrow)$ Will reduce in other dir ...