Undirected case is easier . . .

because $d(\cdot,\cdot)$ is a <u>metric</u>

$$d(u,v) = d(v,u) \quad \text{(symmetry)}$$
$$d(u,v) \le d(u,w) + d(w,v) \quad (\Delta \text{ ineq}).$$

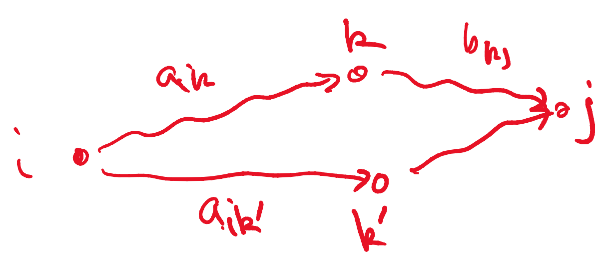
$$\Rightarrow \quad |d(u,v) - d(u,w)| \le d(w,v)$$
$$(\text{“}\Delta \text{ diff. ineq.”})$$

Modified Lemma Given $n \times n$ matrices A, B where all entries of B are in $[c] \cup \{\infty\}$ but all entries of A are arbitrary ints.

s.t. $\forall i, j, k, k'$, if $b_{kj}, b_{k'j} \ne \infty$,
$$|a_{ik} - a_{ik'}| \le 2c.$$

& follows from Δ diff ineq.

Then can compute (min,+)-MM in $\tilde{O}(cn^\omega)$ time.



$$c_{ij} = \min_k (a_{ik} + b_{kj})$$

$$|a_{ik} - a_{ik'}| \le b_{kk'} \le b_{kj} + b_{k'j} \le 2c.$$

Pf:

idea – make entries of A small

i.e. look at $(A \bmod 8c) * B$

(min,+)-MM in $\tilde{O}(cn^\omega)$ time in $\tilde{O}(cn^{\omega})$ time

Consider i, j.

Pick k_0 with $b_{k_0 j} \neq \infty$.

Assume $\underline{(a_{ik_0} \bmod 8c)} \in [3c, 4c]$ if not, shift

Then $(a_{ik} \bmod 8c) \in [c, 6c]$ by assumption

$\forall k$ with $b_{kj} \neq \infty$.

$(a_{ik} \bmod 8c) + b_{kj} \in [c, 7c]$

$(a_{ik} + b_{kj}) \text{ div } 8c = a_{ik_0} \text{ div } 8c$. $\square$

Seidel's Alg'm ('95) APSP in $\tilde{O}(n^{\omega})$ time for undirected unweighted

idea - even vs. odd distances

even case:

Compute $G^2 = (V, E^2)$

$uv \in E^2 \iff \exists w, \underline{uw \in E \wedge wv \in E}$

i.e. take $A \cdot A$ for adj matrix A

Boolean MM. in $O(n^{\omega})$ time

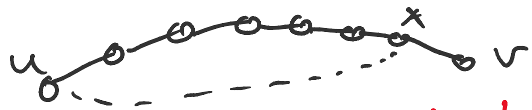
recursively solve APSP on G^2

... all dists by 2 to get D^{even}

recursively solve APSP on G

multiply all dists by 2, to get D^{even}.

odd case:



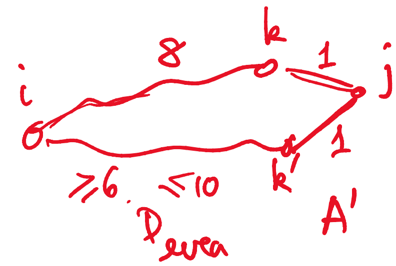
Compute (min,+)-product of $D^{even} * A'$, $a'_{ij} = \begin{cases} 1 & \text{if } a_{ij}=1 \\ \infty & \text{else} \end{cases}$

small values!

large values

(min,+)-prod.

by modified Lemma with $c=1$, $O(n^{\omega})$ time!



$\Rightarrow$ $O(\log n)$ iterations or levels of recursion

$\Rightarrow$ total time $\boxed{O(n^{\omega} \log n)}$
$= \tilde{O}(n^{\omega}) = O(n^{2.373})$

Rmk · Seidel's orig. alg'm avoids mod trick from modified Lemma

Rmk - does not extend to weighted

for small weights in [c]:

Galil-Margalit '97: $\tilde{O}(c^{\frac{\omega+1}{2}} n^{\omega})$

Soshan-Zwick '99: $\tilde{O}(c\, n^{\omega})$

My version of $\tilde{O}(cn^{\omega})$:

idea – back to Zwick's hitting sets

$\delta = 0.1$

Compute $R_{\delta\ell}$ for all $\ell = (3/2)^i$,

$$|R_{\delta\ell}| = \tilde{O}\left(\frac{n}{\ell}\right), \quad R_{\frac{2}{3}\delta\ell} \supseteq R_{\delta\ell}, \quad R_{\delta 1} = V$$

Phase I. for each $\ell = (3/2)^i$ in increas. order:

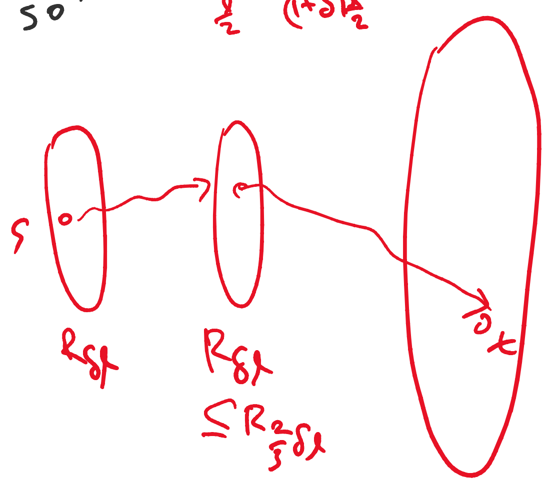
Compute shortest paths from/to $R_{\delta\ell}$ with $\le \ell$ edges; i.e.

compute $d^{(\le \ell)}(s,t)$ $\forall s \in R_{\delta\ell}, t \in V$ (or vice versa)

as follows:

if $\le 2/3\,\ell$ edges, already computed in prev iter.

if $\in (\frac{2}{3}\ell, \ell]$ edges:



$\forall s \in R_{\delta\ell}, t \in V,$

$d^{(\le \ell)} \ldots \le \ldots d^{(\le \frac{2}{3}\ell)}(s,t)$

$\forall x \in R_{\delta\ell}$, ...

$$d^{(\le \ell)}(s,t) = \min \left\{ d^{(\le \frac{2}{3}\ell)}(s,t),\ \min_{x \in R_{\delta\ell}} d^{(\le \frac{2}{3}\ell)}(s,x) + d^{(\le \frac{2}{3}\ell)}(x,t) \right\}$$

from prev iter.

$|R_{\delta\ell}| = \tilde{O}(\frac{n}{\ell})$

$$\Rightarrow \tilde{O}\left(c\ell\, M\left(\frac{n}{\ell}, \frac{n}{\ell}, n\right)\right).$$

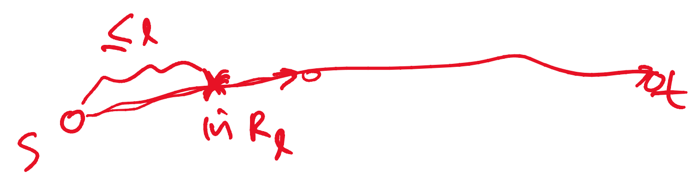
Phase II. for each $\ell = (3/2)^i$ in decreas order:

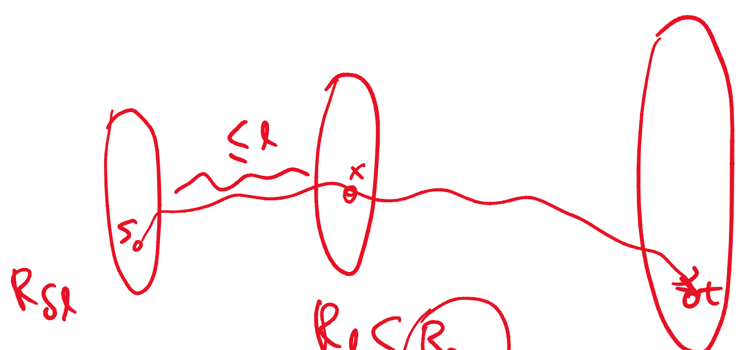
Compute shortest paths from $R_{\delta\ell}$ with no length restrictions.

as follows:

if $\lesssim \ell$ edges, already computed from Phase I.

if $> \ell$ edges,





$R_\ell \subseteq R_{\frac{3}{2}\delta\ell}$

$(\ell \ge \frac{3}{2}\delta\ell)$

$|R_{\delta\ell}| = \tilde{O}(\frac{n}{\ell})$

$|R_\ell| = \tilde{O}(\frac{n}{\ell})$

$\forall s \in R_{\delta\ell}, t \in V,$

$d(s,t) = \min \{ d^{(\le \ell)}(s,t),$ ...

$$d(s,t) = \min \left\{ d^{(S_x)}(s,t),\ \min_{x \in R_\ell} d^{(S_\ell)}(s,x) + d(x,t) \right\}$$

entries in $[c\ell]$

from prev iter.

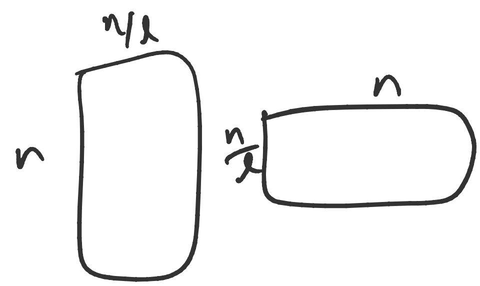
could be big

by Modified Lemma,

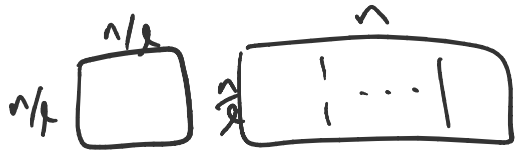
$$\tilde{O}\left(c\ell\, M\left(\tfrac{n}{\ell}, \tfrac{n}{\ell}, n\right)\right)$$

total time: $\tilde{O}\left(\sum_{\ell=(3/2)^i} c\ell\, M\left(\tfrac{n}{\ell}, \tfrac{n}{\ell}, n\right)\right)$

twice



now



Quick upper bd: $\tilde{O}\left(c\ell \cdot \ell \left(\tfrac{n}{\ell}\right)^{\omega}\right)$

$$= \tilde{O}\left(\frac{c n^{\omega}}{\ell^{\omega-2}}\right)$$

Sum over all $\ell = (3/2)^i \Rightarrow \boxed{\tilde{O}(cn^{\omega})}$

What about large int wts or real wts??

What about large int wts or real wts??

OPEN

Vertex wts easier than edge wts ...