September 11, 2020 10:41 AM

- HW1
- Last Time: subset sum

  $\tilde{O}(\sqrt{n}\,t)$ det.

  $\tilde{O}(t)$ rand.

  Cond. LB: $\sim\Omega(t^{1-\delta})$ ...

  generalization: $(\min,+)$ - convol.

  $\sum_i a_i b_{k-i}$

  $\downarrow$

  $\min_i (a_i + b_{k-i})$

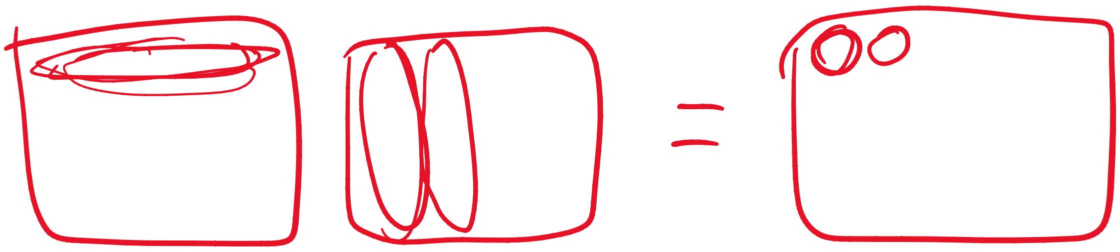
  ⋮

Matrix Multiplication

Problem Given 2 $n \times n$ matrices $A = (a_{ij})$, $B = (b_{ij})$,

Compute $C = AB$

where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$



trivial alg'm: each c_{ij} in $O(n)$ time

$\Rightarrow O(n^2 \cdot n) = \boxed{O(n^3)}$

naive lower bd: $\Omega(n^2)$

Strassen's Alg'm ('69)

$(\quad)(\quad) = (\quad)$

Strassen's Alg'm ('64)

$(\quad)(\quad) = (\quad)$

first consider $n=2$: given $(a_{ij}), (b_{ij})_{\substack{i=1,2\\ j=1,2}}$

want
$$\begin{cases} c_{11} = a_{11}b_{11} + a_{12}b_{21} & c_{12} = a_{11}b_{12} + a_{12}b_{22} \\ c_{21} = a_{21}b_{11} + a_{22}b_{21} & c_{22} = a_{21}b_{12} + a_{22}b_{22} \end{cases}$$

naively, 8 mults

idea - power of subtraction again! (like Karatsuba)

$$\text{7 mults}\left\{\begin{aligned} p_1 &= a_{11}(b_{11} - b_{21}) \\ p_2 &= (a_{11} + a_{12})b_{21} \\ p_3 &= a_{22}(b_{22} - b_{12}) \\ p_4 &= (a_{21} + a_{22})b_{12} \\ p_5 &= (a_{11} + a_{22})(b_{21} + b_{12}) \\ p_6 &= (a_{12} - a_{22})(b_{21} + b_{22}) \\ p_7 &= (a_{21} - a_{11})(b_{11} + b_{12}) \end{aligned}\right.$$

$$c_{11} = p_1 + p_2$$
$$c_{22} = p_3 + p_4$$
$$c_{12} = p_5 + p_6 + p_3 - p_2$$
$$c_{21} = p_5 + p_7 + p_1 - p_4$$

magical!

7 mults!

general n?

idea - D & C.

$$A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) \qquad B = \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right)$$

$$\Rightarrow C = \left(\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right)$$

same formulas but with a_{ij}'s $\to$ A_{ij}'s
b_{ij}'s $\to$ B_{ij}'s

same formulas but with a_{ij}'s $\to$ A_{ij}'s

b_{ij}'s $\to$ B_{ij}'s

$c_{ij} \to C_{ij}$

note: not Commutative ($AB \neq BA$)

by formula, 7 recursive calls

$$\Rightarrow \quad T(n) = 7\,T\left(\frac{n}{2}\right) + O(n^2)$$

$$\Rightarrow \quad O(n^{\log_2 7}) = \boxed{O(n^{2.81})}$$

Better?

History:

idea - 3-way D & C

$$T(n) = \underline{23}\,T\left(\frac{n}{3}\right) + O(n^2)$$

Laderman '76

$$\Rightarrow \quad O(n^{\log_3 23}) = O(n^{2.85})$$

worse!

Pan '78

$$T(n) = 143640\,T\left(\frac{n}{70}\right) + O(n^2)$$

$\frac{k^3-4k}{3} + 6k^2$ $\qquad$ k

$$\Rightarrow \quad O(n^{\log_{70} 143640}) = O(n^{2.796})$$

Pan '79 : $O(n^{2.781})$

Bini et al. '80 : $O(n^{2.780})$

Schönhage '81 : ... $O(n^{2.548})$

... $O(n^{2.522})$

Strassen '86 : $O(n^{2.479})$

→ Coppersmith - Winograd '90 : $O(n^{2.376})$

Stothers '10 $O(n^{2.374})$

Vassilevska-W '12 $O(n^{2.373})$

Le Gall '14 $O(n^{2.372864})$

MAJOR OPEN Q :

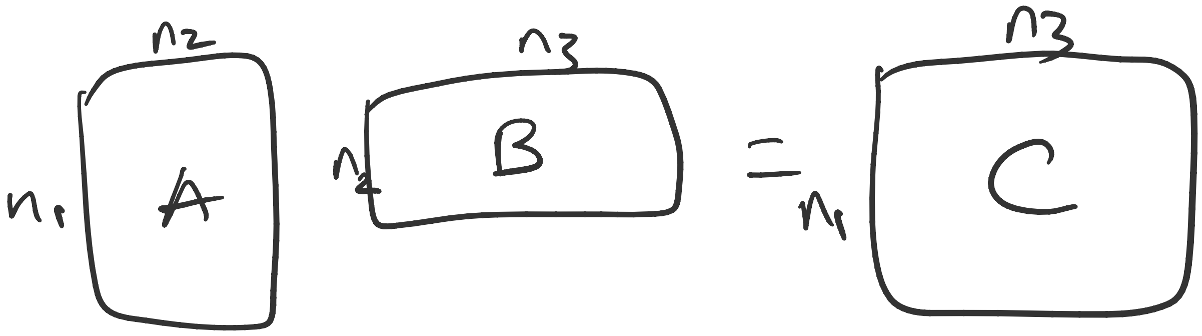
what is best exponent ?

denoted ω

$$2 \leq \omega < 2.372864$$

Rmk - in some appl'ns,

need rectangular matrix multiplication



let $M(n_1, n_2, n_3)$ = time to multiply $n_1 \times n_2$ with $n_2 \times n_3$ matrix

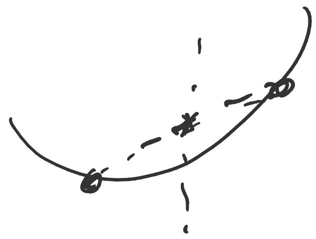
$n_1 \times n_2$ with $n_2 \times n_3$ matrix

$$\omega(a,b,c) = \text{exponent for } M(n^a, n^b, n^c)$$

$$(\omega = \omega(1,1,1))$$

Known Properties:

$\omega(\cdot,\cdot,\cdot)$ is convex.

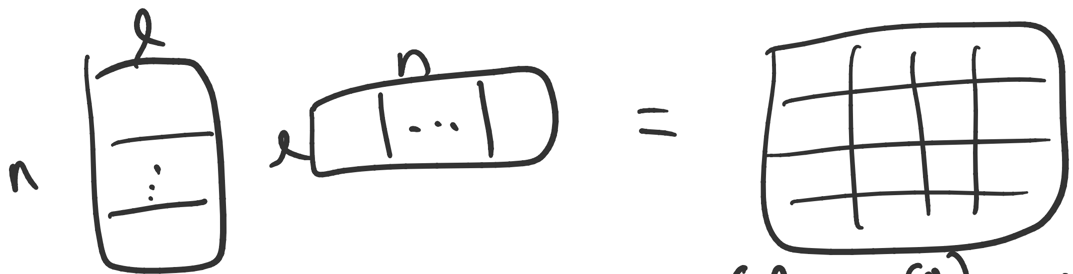


$$\omega(a,b,c) = \omega(c,b,a) = \omega(b,a,c)$$ symmetry

Quick Bds: If $\ell \le n$,

$$M(n,\ell,n) = O\left(\left(\frac{n}{\ell}\right)^2 \cdot \ell^\omega\right) = O(\ell^{\omega-2} n^2).$$

not tight



If $\ell > n$, $M(n,\ell,n) = O\left(\frac{\ell}{n} \cdot n^\omega\right) = O(\ell n^{\omega-1})$

not tight