recurse $\Rightarrow$ partition tree

$S(n) = \boxed{O(n)}$

halfplane / triangle counting:

$$Q(n) \le c\sqrt{t}\, Q\left(\frac{n}{t}\right) + O(t)$$

$$\Rightarrow Q(n) = O\left(n^{\frac{\log c\sqrt{t}}{\log t}}\right)$$

$$= O\left(n^{\frac{1}{2} + \frac{\log c}{\log t}}\right)$$

$$= \boxed{O\left(n^{\frac{1}{2}+\varepsilon}\right)} \text{ for suff. large const } t$$

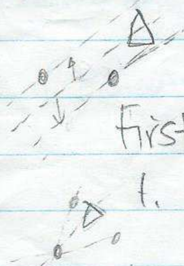
Even better: pick nonconst $t = n^{\varepsilon}$ (# levels $= O(\log\log n)$)

$$\Rightarrow Q(n) = O\left(\sqrt{n}\, c^{O(\log\log n)}\right)$$

$$= \boxed{O\left(\sqrt{n} \log^{O(1)} n\right)}.$$

<u>Proof of Partition Thm</u>

Suffices to prove crossing # for a finite set L of m "test" lines



($m = O(n^2)$ $\xrightarrow{\text{straightforward}}$ can be reduced to $m = O(t)$).

by considering lines thru 2 pts

First Attempt:

1. apply Cutting Lemma to L with $r \approx \sqrt{t}$

   $\Rightarrow$ # cells $= O(r^2) \approx O(t)$

2. subdivide cells to ensure each has $\le \frac{n}{t}$ pts

   $\Rightarrow \le t$ extra vertical cuts

   $\Rightarrow$ # cells is still $O(t)$.

Analysis: total crossings between lines & cells

$$O\Big(\underbrace{t}_{\#\text{ cells}} \cdot \underbrace{m/r}_{\substack{\#\text{ lines}\\ \text{intersecting each cell}}}\Big) = O(m \cdot \sqrt{t})$$

$\Rightarrow$ average # crossings per line $= O(\sqrt{t})$

BUT how to turn average to max??

idea - iterative reweighting (Welzl '88)
(also called multiplicative weight update method,
(occurred in approx alg/m, ML, ...)

Matoušek's Alg/m:

define multiset $\hat{L}$, initialize to L, all multiplicities $= 1$
"weight" / "importance"

for $i = t, \ldots, 1$ do:

// assume in/t pts remaining

1. apply Cutting Lemma to $\hat{L}$ with $r_i \approx \sqrt{i}$
   $\Rightarrow$ # cells $= O(r_i^2) \approx i$
2. pick cell Δ_i with $\geq n/t$ pts.
3. shrink Δ_i to have exactly n/t pts P_i, & remove P_i
4. for each $\ell \in L$ crossing Δ_i
   double multiplicity of ℓ in $\hat{L}$.

so that ℓ will not cross many future cells

Analysis:

before iteration i,

let $m_i = |\hat{L}|$ (multiplicity included)

$e_i = |\{\ell \in \hat{L} : \ell \text{ crosses } \Delta_i\}|$

Know $e_i \leq \frac{m_i}{r_i} \approx \frac{m_i}{\sqrt{i}}$

$$\Rightarrow m_{i-1} \leq m_i + e_i \lesssim \left(1 + \frac{1}{\sqrt{i}}\right) m_i$$

$$\Rightarrow m_0 \leq \left(1 + \frac{1}{\sqrt{1}}\right)\left(1 + \frac{1}{\sqrt{2}}\right) \cdots \left(1 + \frac{1}{\sqrt{t}}\right) m$$

$$= m \prod_{i=1}^{t} \left(1 + \frac{1}{\sqrt{i}}\right)$$

$$\leq m \prod_{i=1}^{t} e^{\frac{1}{\sqrt{i}}} = m e^{\sum_{i=1}^{t} \frac{1}{\sqrt{i}}}$$

$$\approx m e^{O(\sqrt{t})}$$

N9

But $m_{final} = \sum_{\ell \in L} (\text{final multiplicity of } \ell)$

$$= \sum_{\ell \in L} 2^{(\text{crossing \# of } \ell)}$$

$\Rightarrow \forall \ell \in L$,

crossing # of $\ell \le \log_2 m_{final}$

$\le O(\log m + \sqrt{t})$. $\square$

Note - extends to d dims $(O(t^{1-1/d}))$

- $O(n)$ time when t is const

($\varepsilon/10$: avoid const factor blowup in recursion by working on all nodes in one level simultaneously)

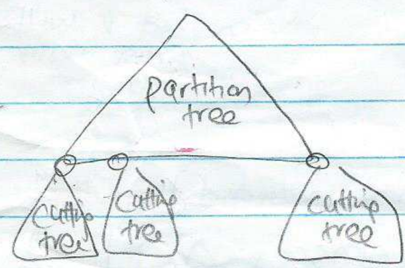
Consequences

1. space/time tradeoff

build partition tree

when # pts $\le s$,

build cutting tree



$\Rightarrow$ space & preproc time $O\left(n + \frac{n}{s} \cdot s^{2+\varepsilon}\right) = \boxed{O^*(m)}$ $\quad m = ns$

(O^* hides n^ε factors)

query time: $O\left(\left(\frac{n}{s}\right)^{\frac{1}{2}+\varepsilon} \log s\right)$

$$= O^*\left(\left(\frac{n}{m/n}\right)^{\frac{1}{2}}\right) = \boxed{O^*\left(\frac{n}{\sqrt{m}}\right)}$$

($n/m^{1/d}$ in d dims)

2. cost of answering n queries for n pts in 2D

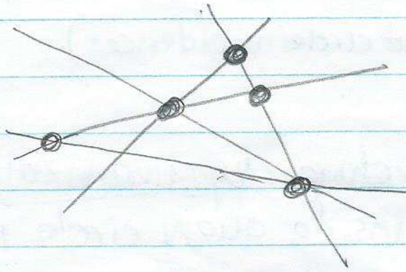
$= O^*\left(m + n \cdot \frac{n}{\sqrt{m}}\right)$

$= \boxed{O^*(n^{4/3})}$ set $m = n^{4/3}$

($n^{2-\frac{2}{d+1}}$ in d dims)

(1983)

3. related to Szemeredi-Trotter Thm from combinatorial geometry:
given n lines & n pts in $\mathbb{R}^2$,
\# pairs (p, ℓ) with p incident on ℓ is $O(n^{4/3})$
tight



5 lines, 5 pts
12 incidences

Pf: $I(n,m)$ = max # incidences for n lines, m pts
Then $I(n,m) = O(n^2 + m)$
sum of degrees in arrangement

[first try - $I(n,n) \le r\, I(\frac{n}{r}, n)$
$= O(r((\frac{n}{r})^2 + n))$ set $r = \sqrt{n}$
$= O(n^{3/2})$]

Apply Cutting Lemma
$\Rightarrow$ $O(r^2)$ cells
Subdivide cells s.t. each has $\le m/r^2$ pts
$\Rightarrow$ still $O(r^2)$ cells

Then $I(n,n) = O(r^2)\ I\left(\frac{n}{r}, \frac{n}{r^2}\right)$

$= O(r^2)\ I\left(\frac{n}{r^2}, \frac{n}{r}\right)$ by duality

$= O(r^2)\left(\left(\frac{n}{r^2}\right)^2 + \frac{n}{r}\right)$

$= O(r^2)\left(\frac{n^2}{r^4} + \frac{n}{r}\right)$ set $r = n^{1/3}$

$= O(n^{4/3})$. $\square$

4. also related to Erdös unit distance Problem:
given n pts in $\mathbb{R}^2$, # pairs of distance exactly 1
is $O(n^{4/3})$

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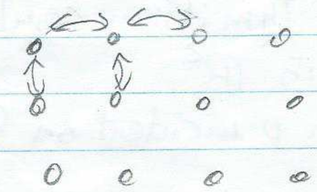
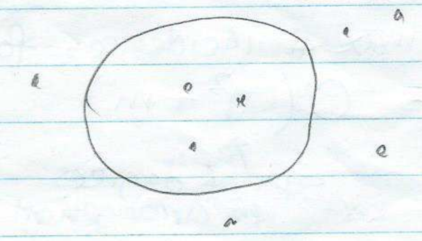


still open! (\$500)
current lower bd:
$\Omega\left(n^{1+c/\log\log n}\right)$.

(reduces to pt-unit circle incidences)

5. nonlinear range searching by <u>linearization</u>

e.g. count # pts inside query circle in 2D



find $(x,y) \in P$ s.t.

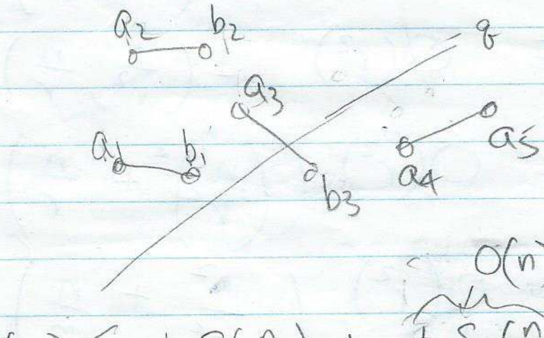
$$\sqrt{(x-q_x)^2 + (y-q_y)^2} \le q_r .$$

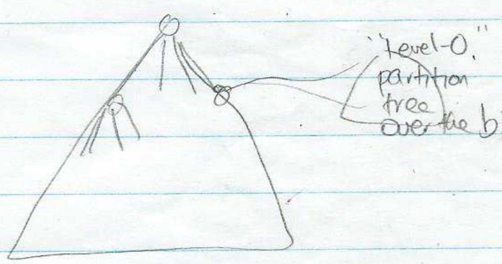
i.e.
$$\underbrace{x^2+y^2}_{z} - 2q_x x - 2q_y y + q_x^2 + q_y^2 - q_r \le 0.$$

reduces to halfspace range counting in 3D.

6. multilevel partition/~~cutting~~ trees

e.g. count # line segments intersecting query line in 2D





$$S(n) \le t\,S\left(\tfrac{n}{t}\right) + \overbrace{t\,S_0\left(\tfrac{n}{t}\right)}^{O(n)}$$

$$\Rightarrow O(n \log n)$$

$$Q(n) \le C\sqrt{t}\,Q\left(\tfrac{n}{t}\right) + \overbrace{t\,Q_0\left(\tfrac{n}{t}\right)}^{O(\sqrt{n})}$$

$$\Rightarrow O\left(n^{\frac{1}{2}+\varepsilon}\right)$$