recurse $\Rightarrow$ partition tree

$S(n) = \Theta(n)$

half-plane/triangle counting:

$Q(n) \leq C \log n + O(1)$

$\Rightarrow Q(n) = O\left( n^{\frac{1}{2} + \log n} \right)$

$= O\left( n^{1/2 + 1/\log \log n} \right)$

for suff. large const $t$

Even better: pick nonconst $t = n^{1/3}$ (# levels = O(loglogn))

$\Rightarrow Q(n) = O\left( n^{1/3 \log \log n} \right)$

$= O\left( n^{1/3 \log \log n} \right)$

Proof of Partition Thm:

Suffices to prove crossing # for a finite set $L$ of "test" lines

$\Delta$ ($m = \Theta(n^2)$ straightforward $\Rightarrow$ can be reduced to $m = O(t)$)

by concurrence lines thru 2 pts

First Attempt:

1. apply Cutting Lemma to $L$ with $r = \sqrt{t}$

$\Rightarrow$ # cells $= O(r^2) = O(t)$

2. subdivide cells to ensure each has $\leq t$ pts

$\Rightarrow$ at most $t$ extra vertical cuts

$\Rightarrow$ # cells is still $O(t)$.

Analysis: total crossings between lines & cells

$O\left( t \cdot \frac{m}{r} \right) = O\left( m \cdot \sqrt{t} \right)$

# cells intersecting each cell

$\Rightarrow$ average # crossings per line $= O(\sqrt{t})$

BUT how to turn average to max??
idea - iterative reweighting (Welzl '88)
(also called multiplicative weight update method,
occurred in approx algm, IML, etc.)

Matoušek's Alg
define multiset \( \hat{L} \), initialize to \( L \), all multiplicities = 1
for \( i = t, \ldots, 1 \) do:
  // assume \( \ln \) pts remaining
  1. apply Cutting Lemma to \( \hat{L} \) with \( r_i = \sqrt{t} \)
     \( \Rightarrow \) # cells \( = O(r_i^2) \approx \ln t \)
  2. pick cell \( \Delta_i \) with \( \geq \frac{\ln}{t} \) pts.
  3. shrink \( \Delta_i \) to have exactly \( \frac{\ln}{t} \) pts \( P_i \), & remove \( P_i \)
  4. for each \( \ell \in L \) crossing \( \Delta_i \)
     double multiplicity of \( \ell \) in \( \hat{L} \)

Analysis:
before iteration \( i \),
let \( m_i = |\hat{L}| \) (multiplicity included)
\( e_i = \{ \ell \in \hat{L}: \ell \text{ crosses } \Delta_i \} \)

Know \( e_i \leq \frac{m_i}{\ln t} \)
\( \Rightarrow m_{i+1} \leq m_i + e_i \leq \left(1 + \frac{1}{\ln t}\right)m_i \)
\( \Rightarrow m_0 \leq \left(1 + \frac{1}{\ln t}\right)^t m \)
\( = m \prod_{i=1}^{t} \left(1 + \frac{1}{\ln t}\right) \approx m e^{\frac{t}{\ln t}} \)
\( \leq m \prod_{i=1}^{t} e^{\frac{1}{\ln t}} = m e^{t \frac{1}{\ln t}} \approx m e O\left(\sqrt{t}\right) \)
But \( m_{\text{final}} = \sum_{l \in L} (\text{final multiplicity of } l) \sum_{l \in L} (\text{crossing # of } l) \Rightarrow \forall l \in L, \text{crossing # of } l \leq \log_2 m_{\text{final}} \leq O(\log m + \sqrt{t}) \).

Note - extends to \( d \) dims \( O(t^{d-1/2}) \) \( O(n) \) time when \( t \) is const

Consequences
1. Space/time tradeoff build partition tree when \( \# \text{ pts} \leq s \), build cutting tree

\[
\Rightarrow \text{space & preprocess time} = O^*(M) \quad m = n^d \quad (O^* \text{ hides } n^1 \text{ factors})
\]

\[
\text{query time} = O^*(\left(\frac{n}{s}\right)^{1/2} + \epsilon \log s) = O^*(\left(\frac{n}{m/n}\right)^{1/2}) = O^*(\frac{n}{\sqrt{m}})
\]

2. Cost of answering \( n \) queries for \( n \) pts in 2D

\[
= O^*(m + n \cdot \frac{n}{\sqrt{m}}) = O^*(n^{4/3}) \quad \text{set } m = \frac{n}{\sqrt{m}}
\]
3. related to Szemerédi-Trotter Thm: from combinatorial geometry:

given $n$ lines & $n$ pts in $\mathbb{R}^2$,

$\#$ pairs $(p,q)$ with $p$ incident on $l$ is $O(n^{4/3})$.

5 lines, 5 pts
12 incidences

**Proof:**

$I(n,m) = \text{max } \# \text{ incidences for } n \text{ lines, } m \text{ pts}$

Then $I(n,m) = O(n^2 + m)$

Apply Cutting Lemma

$\Rightarrow O(2^r)$ cells

Subdivide cells s.t. each has $\leq m/r^2$ pts

$\Rightarrow$ still $O(2^r)$ cells

Then $I(n,n) = O(r^2)I\left(\frac{n}{r}, \frac{r}{r^2}\right)$

$= O(r^2)I\left(\frac{n}{r}, \frac{r}{r^2}\right)$ by duality

$= O(r^2)\left(\frac{(n^2)^2 + n}{r}\right)$

$= O(r^2)\left(\frac{n^3 + n}{r^4}\right)$ set $r = n^{1/3}$

$= O(n^{4/3})$. $\square$

4. also related to Erdős unit distance problem:

given $n$ pts in $\mathbb{R}^2$, $\#$ pairs of distance exactly 1
is $O(n^{4/3})$.
still open! ($500)
current lower bd:
\[
\sum_{i} (1 + \frac{c}{\log \log n})
\]
(reduces to pt-unt circle incidences)

5. nonlinear range searching by linearization
   e.g. count # pts inside query circle in 2D

   \[
   \sqrt{(x-q_x)^2 + (y-q_y)^2} \leq qr
   \]
   i.e. \[
   \frac{x^2 + y^2}{2} - 2q_x - 2q_y + q_x^2 + q_y^2 - qr \leq 0
   \]
   reduces to half-space range counting in 3D.

6. multilevel partition/cutting trees
   e.g. count # line segments intersecting query line in 2D

   \[
   S(n) \leq tS(\frac{n}{t}) + tS_0(\frac{n}{t})
   \]
   \[
   \Rightarrow O(n \log n)
   \]
   \[
   Q(n) \leq CLT Q(n) + tQ_0(n)
   \]
   \[
   \Rightarrow O(n^{t+\epsilon})
   \]