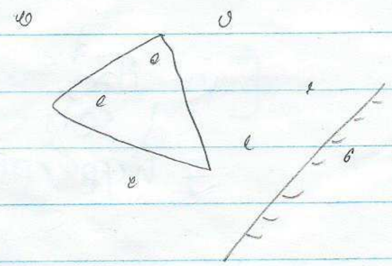
Nonorthogonal Range Searching



triangular range search
(or simplex)

halfplane range search
(or halfspace)

History

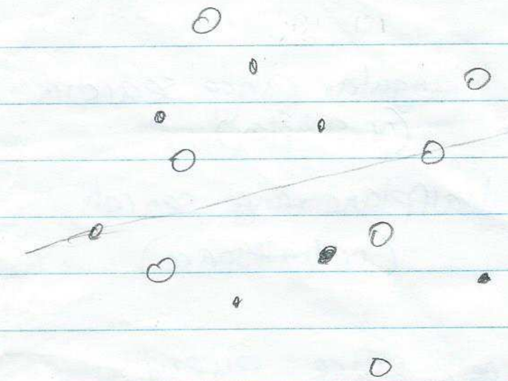
	preproc	space	query	
⇒ Willard '82		n	$n^{0.793}$	(in 3D, $n^{0.936}$)
		n	$n^{0.774}$	$\log_6 4$
Edelsbrunner-Welzl '86		n	$n^{0.695}$	
Haussler-Welzl '88		n	$n^{0.667}$	(rand.)
Welzl '88		n	$\sqrt{n}\log n$	
Chazelle-Sharir-Welzl '90	$n^{1+\varepsilon}$	n	$n^{1/2+\varepsilon}$	
⇒ Matoušek '91	$n\log n$	n	$\sqrt{n}\log^{O(1)} n$	
Matoušek '92	$n^{1+\varepsilon}$	n	$\sqrt{n}$	$n^{1-1/d}$
C. '10	$n\log n$	n	$\sqrt{n}$	(rand.)
⇒ Clarkson '87		$n^{2+\varepsilon}$ & $n^{d+\varepsilon}$	$\log n$	(rand.)
⇒ Matoušek '92		m	$\frac{n}{\sqrt{m}}\log^3 n$	
			$(n \le m \le n^2)$	

Method 1: Willard '82

Ham-Sandwich Cut Thm Given any 2 point sets P, Q in $\mathbb{R}^2$, ∃ line that simultaneously bisects P & Q.

(ham: P, bread: Q)

N1



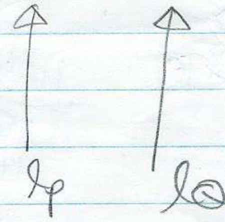
[in $\mathbb{R}^d$,
d sets
$\exists$ hyperplane...]

Pf: Given direction v,

let $\ell_P(v)$ = line bisecting P pointing in dir v

$\ell_Q(v)$ = " " Q " " "

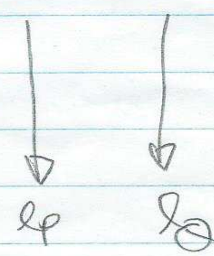
"BEFORE" $\uparrow$ $v=(1,0)$



say ℓ_P left of ℓ_Q

as v rotates, ℓ_P & ℓ_Q move continuously

"AFTER" $\downarrow$ $v=(-1,0)$



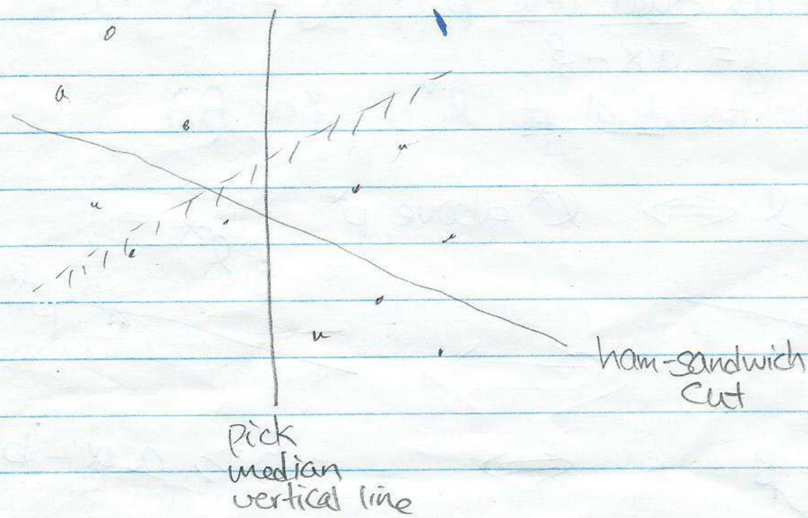
ℓ_P right of ℓ_Q

By intermediate value thm,

$\exists v$ s.t. $\ell_P(v) = \ell_Q(v)$. $\square$

[in $\mathbb{R}^d$, need Borsuk-Ulam Thm from topology]

<u>Cor</u> Given any set P of n pts in $\mathbb{R}^2$,
$\exists$ 2 lines which partition P into 4 subsets of $n/4$ pts



recurse $\Rightarrow$ <u>partition tree</u>

$S(n) = O(n)$

query: given halfplane q,
recurse in 3 cells crossed by line ∂q

$Q(n) = 3\,Q(\frac{n}{4}) + O(1)$

$\Rightarrow O(n^{\log_4 3}) = O(n^{0.793})$ (+k for reporting)

given triangle q,

$Q(n) = O(\#\text{cells crossed by } \partial q)$

$= O(3 \cdot \#\text{cells crossed by a line})$

$= \boxed{O(n^{0.793})}$

Preproc time: $P(n) = 4\,P(\frac{n}{4}) + \underbrace{O(n)}_{\text{Megiddo's ham-sandwich cut alg'm}}$

$\Rightarrow O(n \log n)$

Rmk - 8-sectioning in 3D, but no 16-sectioning in 4D!
[dividing into > 4 parts in 2D ??]

N3

Method 2: (Dual Version)

Def (Duality)

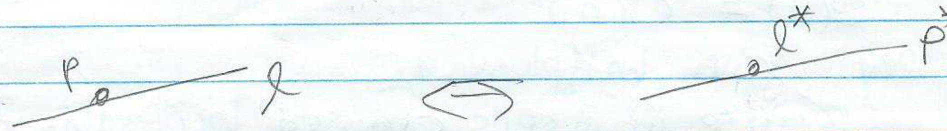
Given pt $p=(a,b)$,
  define its dual line p^*: $y = ax - b$

Given line $y = \alpha x - \beta$,
  define its dual pt ℓ^*: (α, β)

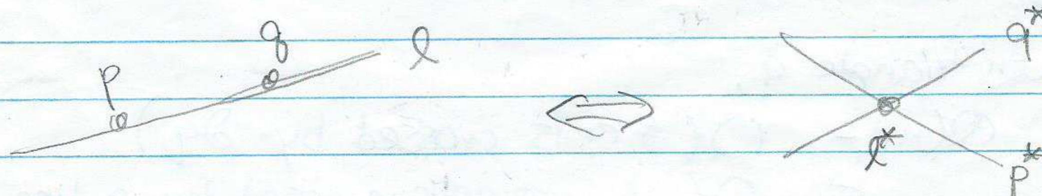
Facts (i) p above $\ell \iff \ell^*$ above p^*

$b \geq \alpha a - \beta \iff \beta \geq a\alpha - b$

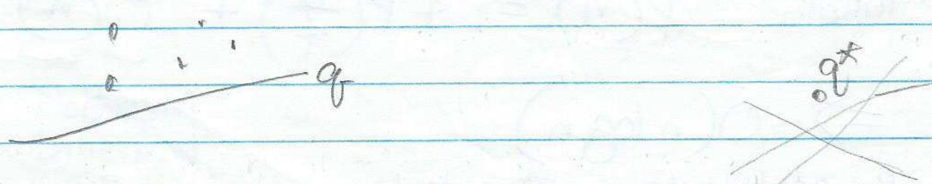
(ii) p is on $\ell \iff \ell^*$ is on p^*



(iii) ℓ is line thru $p, q \iff \ell^*$ is intersection of p^*, q^*

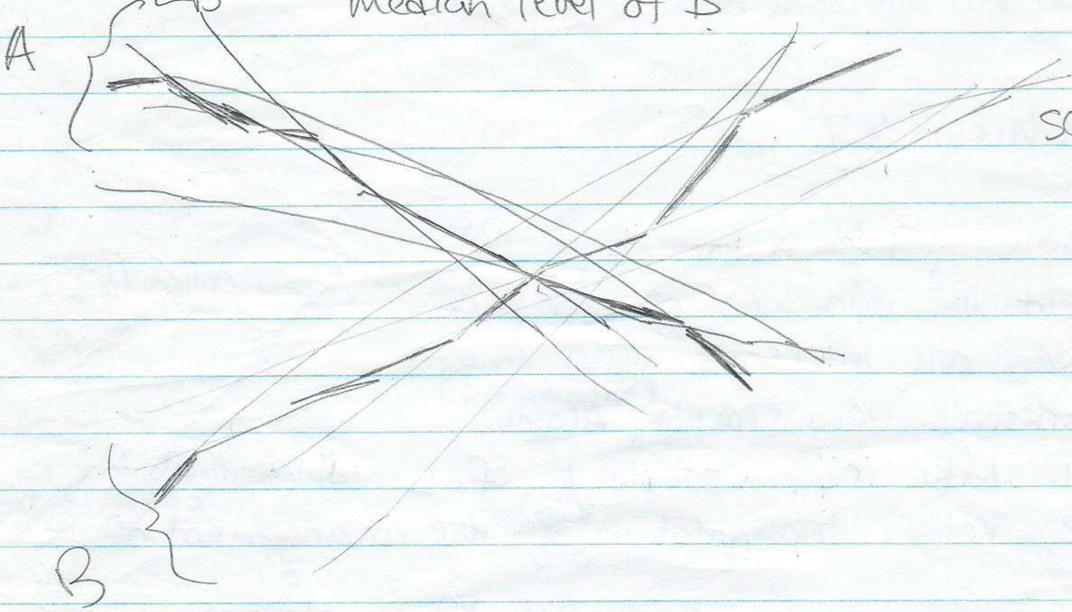


(iv) given n pts, count pts above query line $\iff$ given n lines, count # lines below query p^*



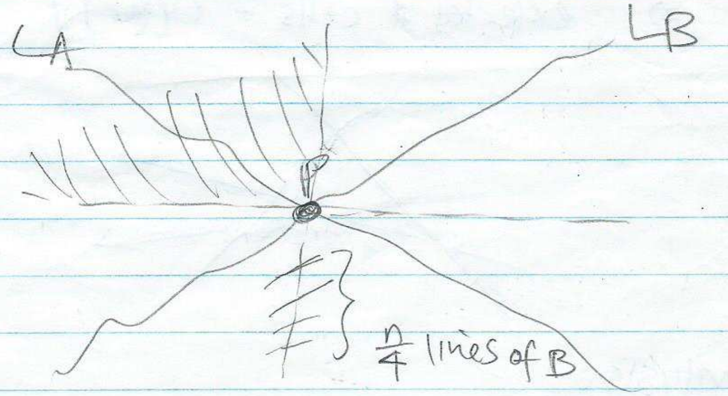
Lemma Given n lines in $\mathbb{R}^2$,
can cut $\mathbb{R}^2$ into 4 cells s.t. each cell intersects $\leq \frac{3n}{4}$ lines

Pf: let m = median slope
A = all lines w. slope $\leq m$
B = all lines w. slope $> m$
$|A| = |B| = n/2$
L_A = median-level of A
(all pts that have half the lines below it)
L_B = median level of B



say $m=0$

in general,



L_A & L_B meet in exactly 1 pt 1 pt p.

draw vertical line at p
& line of slope m at p $\square$

[Rmk- connection w. Ham-Sandwich Cut pf]

NS

recurse $\Rightarrow$ cutting tree

$$S(n) = 4\,S\left(\tfrac{3}{4}n\right) + O(1) \Rightarrow O\left(n^{\frac{\log 4}{\log 4/3}}\right) = \boxed{O\left(n^{4.82}\right)}$$

halfplane counting query:

$$Q(n) = Q\left(\tfrac{3}{4}n\right) + O(1) \Rightarrow \boxed{O(\log n)}$$

[in 2D, simpler method for halfspace counting w. $O(n^2)$ space, $O(\log n)$ time by PL in arrangement but this approach is more general]

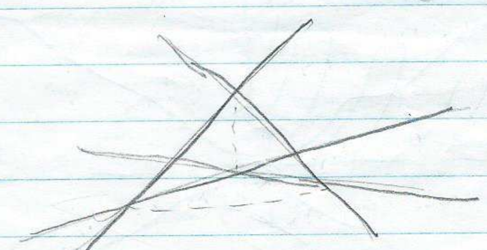
Method 3: Clarkson '87

Cutting Lemma Given n lines L in $\mathbb{R}^2$, can cut $\mathbb{R}^2$ into $O(r^2 \log^2 r)$ cells s.t. each cell intersects $\leq \frac{n}{r}$ lines $\leftarrow$ called $(1/r)$-cutting

Pf of cutting lemma: Clarkson's Very Simple (Randomized) Alg'm:

1. take random sample R of L where each line is chosen w. prob $\frac{br}{n}$.
2. return triangulation $T(R)$ of the arrangement of R

$\Rightarrow$ expected # cells $= O((br)^2)$. $\Leftarrow$ planar graph of size $O((br)^2)$



Analysis:

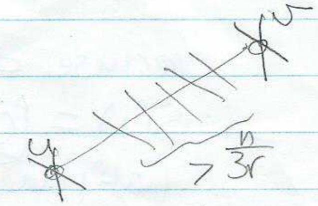
$$\Pr[\text{alg'm errs}] \leq \Pr[\text{some cell of } T(R) \text{ intersects} > \tfrac{n}{r} \text{ lines}]$$
$$\leq \Pr[\text{some edge of } T(R) \text{ intersects} > \tfrac{4n}{3r} \text{ lines}]$$

recurse $\Rightarrow$ cutting tree

$$S(n) = (Cr^2 \log^2 r)\,S\left(\tfrac{n}{r}\right) + O(r^2 \log^2 r) \Rightarrow O\left(n^{\frac{\log(Cr^2 \log^2 r)}{\log r}}\right) = O\left(n^{2 + \frac{\log C}{\log r} + \frac{2\log\log r}{\log r}}\right) = O\left(n^{2+\epsilon}\right) \text{ for suff large const } r$$

halfplane counting: $Q(n) = Q\left(\frac{n}{r}\right) + O(r^2) \Rightarrow \boxed{O(\log n)}$

Fix line seg uv that intersects $> \frac{n}{3r}$ lines



$$\Pr[uv \text{ is an edge of } T(R)] \leq \left(\frac{Cr}{n}\right)^4 \left(1 - \frac{Cr}{n}\right)^{\frac{n}{3r}}$$

\# choices of u, v

$$\Rightarrow \Pr[\text{alg'm errs}] \leq n^4 \cdot \left(\frac{Cr}{n}\right)^4 \left(1 - \frac{Cr}{n}\right)^{\frac{n}{3r}}$$

$$= (Cr)^4 \left(e^{-\frac{Cr}{n}}\right)^{\frac{n}{3r}}$$

$$= \frac{(Cr)^4}{e^{C/3}} \qquad \text{set } C = 15 \ln r$$

$$= O\left(\frac{(\ln r)^4}{r}\right) \ll 1. \quad \square$$

Rmks - the probabilistic method proves existence by using probabilites

- extends to d dims & to very general settings $(O(r^d))$ (related to ε-nets)
- can eliminate extra $\log r$ factors (Chazelle-Friedman '89)
- derandomization possible

Final Method: Matoušek '91

back to primal...

Partition Thm Given n pts P in $\mathbb{R}^2$,

can partition into t subsets $P_1, \ldots, P_t$ each with $\sim \frac{n}{t}$ pts & find t cells $\Delta_1, \ldots, \Delta_t$ with $P_i \subseteq \Delta$.

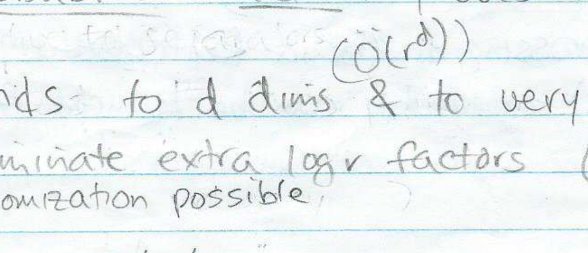
s.t. any line crosses $O(\sqrt{t})$ cells

k-d trees: axis-parallel lines

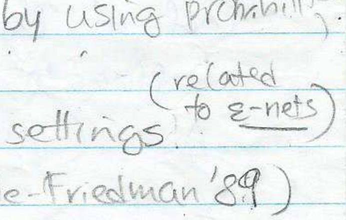
e.g. $t = \frac{n}{2}$

matching w/ low crossing \#

& spanning tree



Bad Ex:



$2\sqrt{t}$ lines

total \# crosses $\geq \Omega(t)$ N