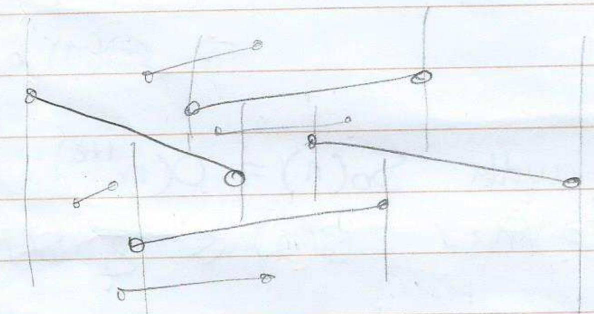
Method 7: Random Sampling (Clarkson-Shor '89)

Cutting Lemma Given n disjoint line segments S, & r,
can divide $\mathbb{R}^2$ into $O(r)$ cells
s.t. each cell intersects $O(\frac{n}{r}\cancel{\log r})$ segs

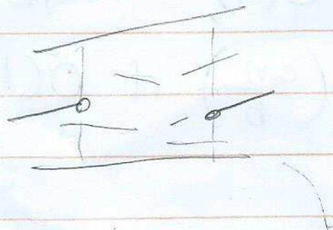
Pf:
1. take random sample $R \subseteq S$ of r segs.
2. return trapezoidal decomposition $T(R)$



Analysis:
$$\Pr[\text{alg'm errs}] \leq \Pr[\text{some cell of } T(R) \text{ intersects} > C\frac{n}{r} \text{ segs}]$$

for each cell Δ, let S_Δ = all segs crossing Δ

D_Δ = all segs touching $\partial\Delta$ ($|D_\Delta| \le 4$)



Fix trapezoid Δ with $|S_\Delta| > C\frac{n}{r}$, $|D_\Delta| = d$.

$Pr[\Delta \in T(R)]$ = Pr[all segs of D_Δ are in R & all segs of S_Δ are not in R)

$$= \left(\frac{r}{n}\right)^d \left(1-\frac{r}{n}\right)^{|S_\Delta|}$$

$$\le \left(\frac{r}{n}\right)^d \left(1-\frac{r}{n}\right)^{C\frac{n}{r}}$$

$$\le \left(\frac{r}{n}\right)^d e^{-C}$$

$$Pr[\text{alg'm errs}) \le \sum_{d\le 4} n^d \cdot \left(\frac{r}{n}\right)^d e^{-C}$$

$$\le r^4 \cdot e^{-C} \ll 1$$

by setting $C = 100 \ln r$. $\square$

(Rmk: to remove log r factor, re-sample in each trapezoid ...)

Apply Lemma with $r = \frac{n}{b}$.

Build DS for $T(R)$.

Build DS inside each cell of $T(R)$ recursively.

$$S(n) \leq \sum_i S(n_i) + S_0\left(\frac{c_0 n}{b}\right) + O(n)$$

with $\sum n_i = O(n)$, $n_i \leq b$.

$$Q(n) \leq Q(b) + Q_0\left(\frac{c_0 n}{b}\right) + O(1)$$

like before

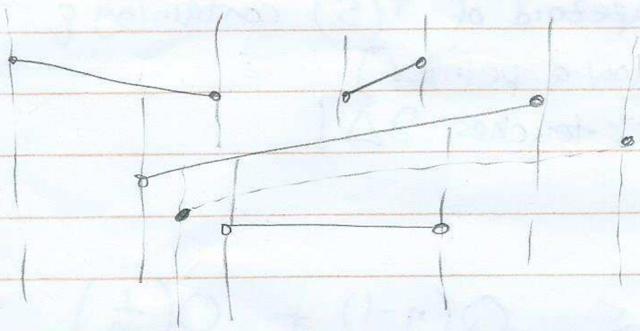
$\Rightarrow$ $\boxed{O(n)}$ space

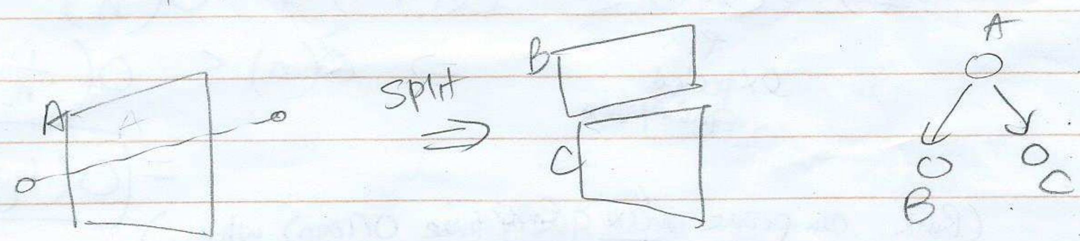
$\boxed{O(\log n)}$ query time

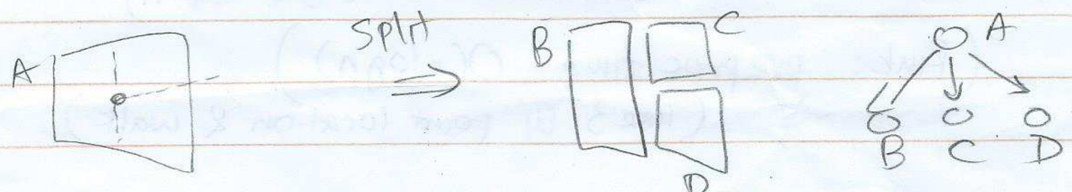
Method 8: Randomized Incremental Construction (Clarkson-Shor '89, Mulmuley)

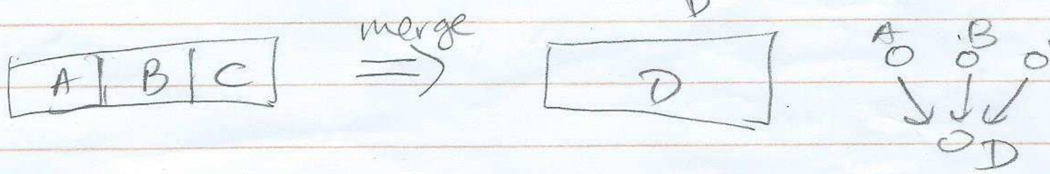
to compute $T(S)$: simplification by choosing $r=n-1$

1. pick rand $s_i \in S$.
2. compute $T(S-\{s_i\})$ recursively
3. add s_i by splitting & merging trapezoids









query - follow pointers in the "history dag"

Space: line 3 $O(\underbrace{\#\text{trapezoids } \Delta \in T(S) \text{ s.t. } s_i \text{ touches } \partial\Delta}_{deg(s_i)})$

$$\sum deg(s_i) = O(n) \Rightarrow E[deg(s_i)] = O(1).$$

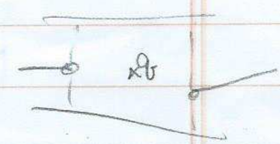
expected space

$$\Rightarrow S(n) \le S(n-1) + O(1)$$
$$\Rightarrow S(n) = \boxed{O(n)}$$

Query analysis: ("oblivious")

assume query pt q is indep of rand. choices

let Δ be trapezoid of $T(S)$ containing q.



Pr[need to follow a pointer]

$= Pr[\, s_i \text{ touches } \partial\Delta \,]$

$\le \frac{4}{n}$

$$\Rightarrow \underset{\text{expected query time}}{Q(n)} \le Q(n-1) + O\left(\frac{1}{n}\right)$$

$$\Rightarrow Q(n) \le O\left(\frac{1}{n} + \frac{1}{n-1} + \dots + 1\right)$$ (Harmonic #)

$$= \boxed{O(\log n)}$$

(Rmk: can prove max query time $O(\log n)$ w.h.p. ...)

(Rmk: preproc time $O(n \log n)$)

(line 3 by point location & walk)