[can be reduce to $O(\frac{1}{\varepsilon^{d-1}})$]

$\Rightarrow$ size $\boxed{O\left(\frac{1}{\varepsilon^d}\right)}$, runtime $O\left(n + \frac{1}{\varepsilon^d}\right)$

Correctness: $\forall$ unit vector v,

$$\omega_v(S) \geq \omega_v(P) - 2\sqrt{d}\,\varepsilon$$
$$\geq (1 - O(\varepsilon))\, \omega_v(P) \quad \text{(by fatness)}$$

ε-Kernel Alg'm 2: idea - rounding dirs

1. make P fat in $[-1,1]^d$
2. form set V_δ of $O\left(\frac{1}{\varepsilon^{d-1}}\right)$ dirs. with $\delta = \Theta(\varepsilon)$
3. for each $v \in V_\delta$

   put extreme pt p_v maximizing $p_v \cdot v$ in S

   q_v minimizing $q_v \cdot v$ in S

$\Rightarrow$ size $\boxed{O\left(\frac{1}{\varepsilon^{d-1}}\right)}$, runtime $O\left(\frac{1}{\varepsilon^{d-1}} n\right)$

Correctness: $\forall$ unit vector u,

say $\omega_u(P) = (p - q) \cdot u \quad (p, q \in P)$

let $v \in V_\delta$ s.t. $\angle(u, v) \leq \delta \quad \Rightarrow \|u - v\| \leq O(\delta)$

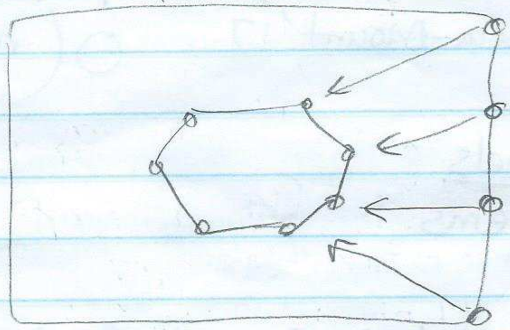
normalized to unit vector

$$\omega_u(S) \geq (p_v - q_v) \cdot u$$
$$\geq (p_v - q_v) \cdot v - O(\delta)$$
$$\geq (p - q) \cdot v - O(\delta)$$
$$\geq (p - q) \cdot u - O(\delta)$$
$$= \omega_u(P) - O(\delta)$$
$$\geq (1 - O(\delta))\, \omega_u(P)$$

Set $\delta = \Theta(\varepsilon)$

ε-Kernel Alg'm 3: Dudley's technique ('74) for polytope approx.

1. make P fat in $[-1,1]^d$
2. form grid G_δ on bdry of $[-2,2]^d$ with side length δ with $\delta = \Theta(\sqrt{\varepsilon})$
3. for each grid pt $g \in G_\delta$, put nearest pt $p_g \in P$ to g in S, $q_g \in P$ to $-g$



$\Rightarrow$ size $O\left(\frac{1}{\delta^{d-1}}\right) = \boxed{O\left(\frac{1}{\varepsilon^{\frac{d-1}{2}}}\right)}$

runtime $O\left(\frac{1}{\varepsilon^{\frac{d-1}{2}}} n\right)$

Correctness: $\forall$ unit vector v,

say $\omega_v(P) = (p-q)\cdot v$

let $g \in G_\delta$ s.t. $\angle(g-p, v) \leq \delta$

$$
\begin{aligned}
p_g \cdot v &= g \cdot v - (g - p_g)\cdot v \\
&\geq g\cdot v - \|g - p_g\|\,\|v\| \\
&\geq g\cdot v - \|g-p\|\,\|v\| \\
&\geq g\cdot v - \frac{(g-p)\cdot v}{\cos\delta} \\
&= g\cdot v - (g-p)\cdot v - O(\delta^2) \\
&= p\cdot v - O(\delta^2)
\end{aligned}
$$

Set $\delta = \sqrt{\varepsilon}$

Similarly, $-q_g \cdot v \geq -q\cdot v - O(\delta^2)$

$\Rightarrow w_u(S) \geq (p_g - q_g)\cdot v \geq (p-q)\cdot v - O(\delta^2) \geq (1 - O(\delta^2))\, w_v(P)$. /7

Improving runtime: like diameter

Combine $\Rightarrow O\left(n + \frac{1}{\varepsilon^{3(d-1)/2}}\right)$

('04) recurse in dim $\Rightarrow O\left(n + \frac{1}{\varepsilon^{d-(3/2)}}\right)$
("discrete Voronoi diagram")

C'17: Chebyshev poly. + dyn prog.

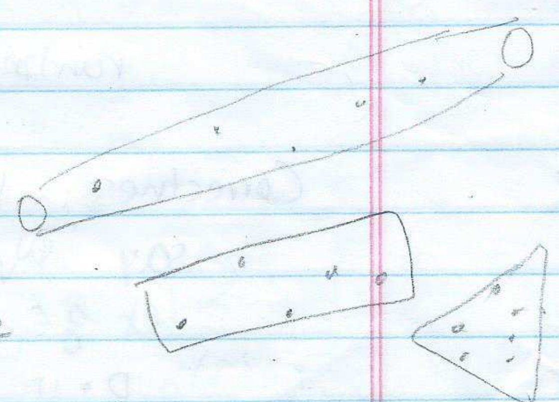
Arya, da Fonseca-Mount'17: $O\left(n \log\frac{1}{\varepsilon} + \frac{1}{\varepsilon^{\frac{d-1}{2}+\alpha}}\right)$
for any const $\alpha > 0$

Applications of ε-Kernels

1. CH-Related Problems

Def $S \subseteq P$ is an ε-coreset for function f if
$$f(P) \geq f(S) \geq (1-\varepsilon) f(P)$$

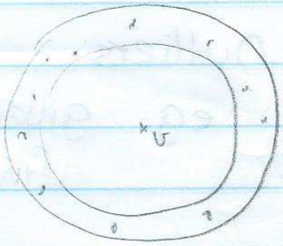
Fact If S is an ε-kernel of P,
then S is an ε-coreset for
min enclos. ball radius
min enclos. cylinder radius
min enclos. box volume
min enclos. simplex volume



More generally, if S is ε-kernel,
$$CH(S) \subseteq CH(P) \subseteq (1+O(\varepsilon))\, CH(S).$$

2. Some Non-convex Problem

Ex min-width enclos. annulus



minimize

$$\max_{p\in P} d(p,v) - \min_{q\in P} d(q,v)$$

over all $v \in \mathbb{R}^2$

where $d(p,v)^2 = (p_1-v_1)^2 + (p_2-v_2)^2$

$= p_1^2 + p_2^2 - 2p_1v_1 - 2p_2v_2 + v_1^2 + v_2^2$

technique - linearization

map $p = (p_1,p_2) \in \mathbb{R}^2$ to $\varphi(p) = (p_1^2+p_2^2, p_1, p_2, 1) \in \mathbb{R}^4$

$v = (v_1,v_2) \in \mathbb{R}^2$ to $\psi(v) = (1, -2v_1, -2v_2, v_1^2+v_2^2) \in \mathbb{R}^4$

$\Rightarrow \quad d(p,v)^2 = \varphi(p)\cdot\psi(v)$.

(pf: $b+a \le B+A$, $\sqrt{B^2-A^2} \le B+A+\sqrt{\delta(b^2-a^2)}$ ... $\le (1+\sqrt{\delta})(B+A)$ ... divide $\Rightarrow B-A \le (1+\sqrt{\delta})(b-a)$)

Easy Math Lemma. Let $0 \le A \le a \le b \le B$.

If $B^2 - A^2 \le (1+\delta)(b^2-a^2)$

then $B - A \le (1+O(\sqrt{\delta}))(b-a)$.

$\Rightarrow$ Fact If $\varphi(S)$ is a δ-kernel of $\varphi(P)$,

then S is an $O(\sqrt{\delta})$-coreset of P

for min enclos. annulus width

[set $\delta = \varepsilon^2$]

[open Q: best ε-dependency?]

3. Small-space Sketches & Streaming

Fact (i) If S_1 is an ε-kernel of P_1

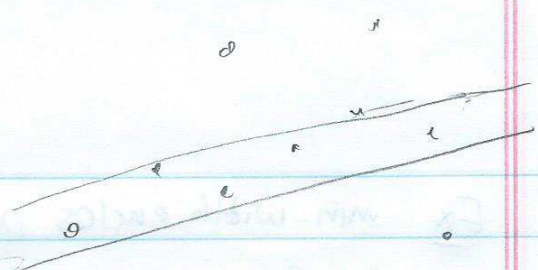
S_2 " " " P_2,

then $S_1 \cup S_2$ is an ε-kernel of $P_1 \cup P_2$.

(requires slight change in Def) (ii) If S is an ε-kernel of P

S' " " ε'-kernel of S,

then S' is an $(\varepsilon+\varepsilon')$-kernel of P.



4. Outliers

eg. given small k

find min-width slab enclosing all but k pts

(or min-volume box
min-radius disk/ball ...)

<u>Fact</u> (Agarwal, Har-Peled, Yu '08)

for $i = 1$ to $2k+1$

S_i = ε-kernel of P ("peeling")

remove S_i from P

Then $\bigcup S_i$ is an ε-coreset of the k-outlier problem

of size $O\left(\frac{k}{\varepsilon^{(d-1)/2}}\right)$.