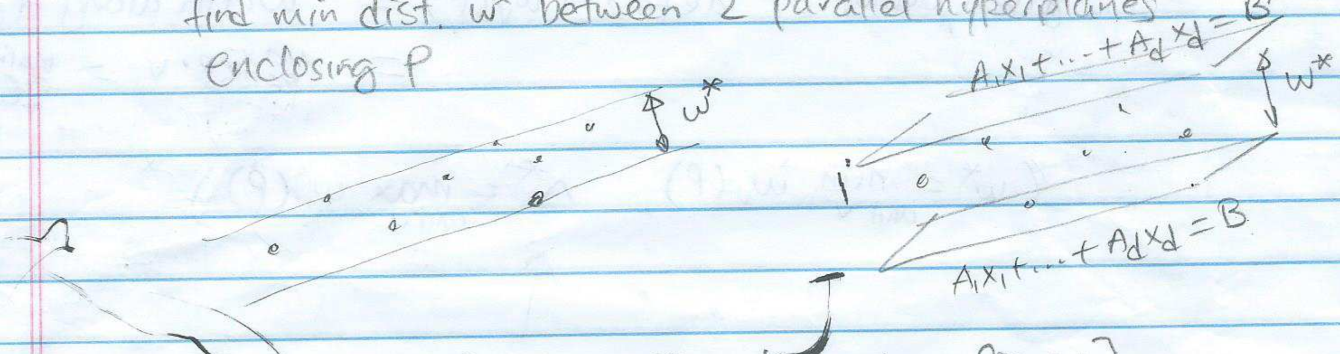
The Width Problem

Given n pts $P \subset \mathbb{R}^d$,

find min dist. w^* between 2 parallel hyperplanes enclosing P



[appl- line fitting in 2D, hyperplane fitting]

Known exact algms:

$d=2$ $\quad O(n \log n)$

$d=3$ $\quad \tilde{O}(n^{3/2})$ (complicated!)

$d \geq 4$ $\quad O(n^{\lceil d/2 \rceil})$

(can make $B' = B+1$)

$$w^* = \min \frac{B' - B}{\sqrt{A_1^2 + \dots + A_d^2}}$$

s.t. $\forall p = (p_1, \dots, p_d) \in P$,

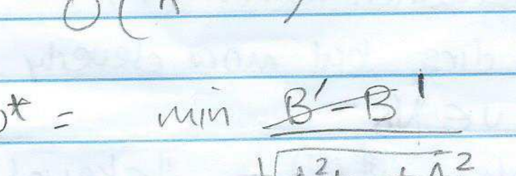
$$B \leq A_1p_1 + \dots + A_dp_d \leq \cancel{B'}\ B+1$$

over vars $A_1, \dots, A_d, B, \cancel{B'}$

intersection of $O(n)$ halfspaces in $d+1$ vars

(a convex polytope)

~~ideas~~ rounding pts



fail!

error $O(\varepsilon\Delta^*)$, may be bigger than $O(\varepsilon w^*)$

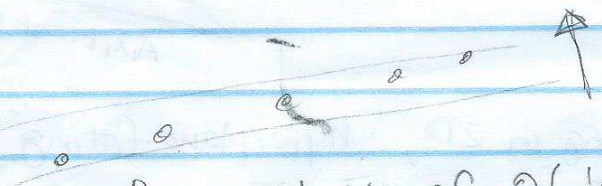
(also called "extent")

- rounding dirs

given unit vector v, define $w_v(P)$ = "width along dir v"

$= \max_{p \in P} p \cdot v - \min_{q \in P} q \cdot v$

$\left(w^* = \min_{\text{unit } v} w_v(P), \quad \Delta^* = \max_{\text{unit } v} w_v(P) \right)$



form set V_δ of $O(1/\delta^{d-1})$ dirs

try all $v \in V_\delta$ $\Rightarrow$ error $O(\delta \Delta^*)$, not $O(\delta w^*)$

fail!

<u>Alg'm 1</u> (Duncan-Goodrich-Ramos '97)

idea - round dirs, but more cleverly

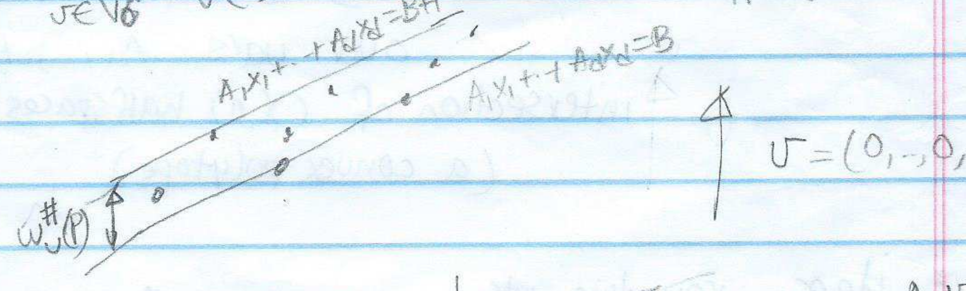
for each $v \in V_\delta$,

compute $w^\#_v(P)$ = "<u>skewed</u> width along dir v"

= min distance measured along v between 2 hyperplanes enclos. P.

return $\min_{v \in V_\delta} w^\#_v(P)$.

e.g.



$$w^\#_v(P) = \min \cancel{\frac{1}{A_1v_1 + \cdots + A_dv_d}} \qquad \max A_1v_1 + \cdots + A_dv_d$$

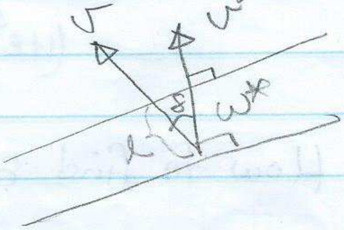
s.t. $\forall p \in P$, $B \le A_1p_1 + \cdots + A_dp_d \le B+1$.

this is <u>linear programming</u> in $d+1$ vars

$\Rightarrow$ $O(n)$ time by known alg'ms

Total time $O(|V_\delta| n) = O\left(\frac{1}{\delta^{d-1}} n\right) = \boxed{O\left(\frac{1}{\varepsilon^{\frac{d-1}{2}}} n\right)}$

Approx factor:



let v^* be optimal dir

let $v \in V_\delta$ s.t. $\angle(v, v^*) \le \delta$

$\Rightarrow w_v^\#(P) \le l \le \frac{w^*}{\cos\delta}$

$\Rightarrow$ factor $\frac{1}{\cos\delta} = 1+\varepsilon$ by setting $\delta = \Theta(\sqrt{\varepsilon})$

<u>Alg'm 2</u>: (Agarwal-Har-Peled-Varadarajan '04)

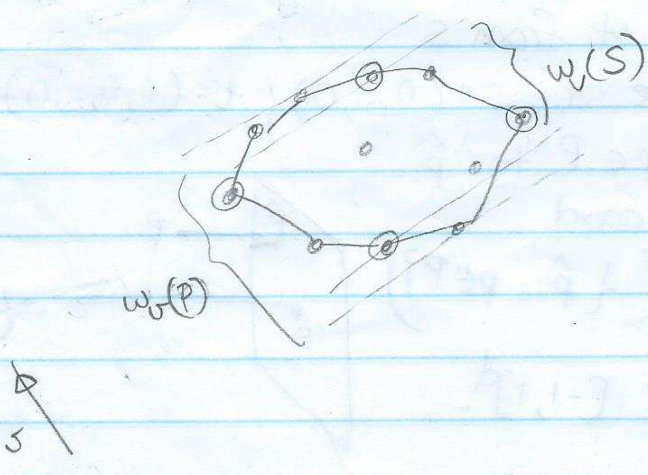
solves more general problem

of approximating width <u>along all dirs</u>

<u>Def</u> An <u>ε-kernel</u> (also called <u>ε-coreset</u> for directional width)

is a subset $S \subseteq P$ s.t.

$\forall v \in \mathbb{R}^d, \quad w_v(S) \ge (1-\varepsilon)\, w_v(P)$



related to "approx convex hull"

<u>Main Thm</u> $\exists$ ε-kernel of size $O\left(\frac{1}{\varepsilon^{(d-1)/2}}\right)$

[optimal]