Alg'm 6 (C. '17)

idea - algebraic approach!

By grid rounding & rescaling,

assume coords are in $\{0, 1, \ldots, U\}$ with $U = \Theta(\frac{1}{\varepsilon})$

Suffice to solve approx decision problem: given r,

decide if $(\Delta^*)^2 \geq r$ or $(\Delta^*)^2 \leq (1+\varepsilon) r$.

Def Chebyshev polynomials

$\cos(k\theta) = T_k(\cos\theta)$

$$T_2(x) = 2x^2 - 1$$

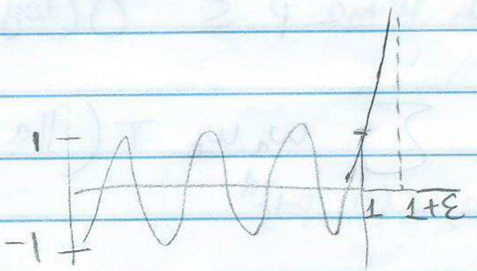
$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

$$T_k(x) = 2x\, T_{k-1}(x) - T_{k-2}(x) = \begin{cases} \cos(k \arccos x) & \text{if } |x| \leq 1 \\ \cosh(k \operatorname{arccosh} x) & \text{if } |x| > 1 \end{cases}$$

Facts $0 \leq x \leq 1 \Rightarrow |T_k(x)| \leq 1$

$x \geq 1+\varepsilon \Rightarrow T_k(x) \geq \cosh(k\sqrt{\varepsilon}) = \frac{e^{k\sqrt{\varepsilon}} + e^{-k\sqrt{\varepsilon}}}{2}$



($\operatorname{arccosh}(1+\varepsilon) \sim \sqrt{\varepsilon}$)

Let $T(x) = M\, T_k\!\left(\frac{x}{r}\right)$ with $M = r^k$

Then $0 \leq x \leq r \Rightarrow |T(x)| \leq M$

$x > (1+\varepsilon) r \Rightarrow T(x) \geq M \cdot \frac{e^{k\sqrt{\varepsilon}}}{2}$

let $Z = \sum_{a,b \in P} T(\|a-b\|^2)$

Then $(\Delta^*)^2 \leq r \Rightarrow Z \leq M \cdot n^2$

$(\Delta^*)^2 > (1+\varepsilon) r \Rightarrow Z > M\left(\frac{e^{k\sqrt{\varepsilon}}}{2} - n^2\right)$

$> M n^2$

by setting $k = \frac{100}{\sqrt{\varepsilon}} \ln n = \Theta\left(\frac{1}{\sqrt{\varepsilon}} \log \frac{1}{\varepsilon}\right)$

$\left(\text{since } n \leq \left(\frac{1}{\varepsilon}\right)^d\right)$

Suffice to compute Z.

Note $|Z| \leq n^2 \cdot r^k (dU^2)^k =: L$.

By Chinese remainder thm,

Suffice to compute $Z \bmod p$

for a few small primes p whose product exceeds M

$\#$ primes $\leq O(\log L) = O(k \log U)$

$O\left(\frac{1}{\sqrt{\varepsilon}} \log^2 \frac{1}{\varepsilon}\right)$

each prime $p \leq O(\log L) = O\left(\frac{1}{\sqrt{\varepsilon}} \log^2 \frac{1}{\varepsilon}\right)$.

$Z \bmod p = \sum_{a,b \in \{0,..,p-1\}^d} w_a w_b T(\|a-b\|^2)$ where $w_a = \#$ pts $q \in P$ with $q \bmod p = a$

$= \sum_{a,b \in \{0,..,p-1\}^d} w_a w_b f_{a-b}$ where $f_s = T(\|s\|^2)$

$= \sum_{a \in \{0,..,p-1\}^d} w_a g_a$ where $g_a = \underbrace{\sum_{b \in \{0,..,p-1\}^d} w_b f_{a-b}}$

Fact can compute g_a for all $a \in \{0, .., p-1\}^d$
in $O(p^d \log p)$ arithmetic ops
by convolution / FFT.

total time $O\left(\left(n + \left(\frac{1}{\sqrt{\varepsilon}} \log^2 \frac{1}{\varepsilon}\right)^d \log \frac{1}{\varepsilon}\right) \cdot \frac{1}{\sqrt{\varepsilon}} \log^2 \frac{1}{\varepsilon}\right)$

$\cdot \log^2 \frac{1}{\varepsilon}$

arithmetic op

$= \boxed{O\left(\left(\frac{1}{\sqrt{\varepsilon}} n + \frac{1}{\varepsilon^{(d+1)/2}}\right) \log^{O(1)} \frac{1}{\varepsilon}\right)}$

Rmk: C'17 has another alg'm without FFT
Arya, da Fonseca, Mount '17

$O\left(n \log \frac{1}{\varepsilon} + \frac{1}{\varepsilon^{(d-1)/2 + \alpha}}\right)$ for any const $\alpha > 0$

(complicated)

Open: better than $n + \frac{1}{\varepsilon^{(d-1)/2}}$?

Har-Peled '01: experimiental results.