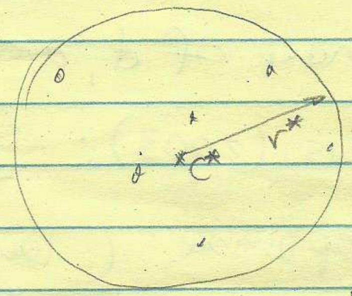
High-Dimensional Problems:

Min Enclosing Ball (MEB / 1-Center)



given n pts $P \subset \mathbb{R}^d$

Exact alg'ms:

"LP-type" problem

$\Rightarrow$ $O(n)$ time for const d

more precisely,

$O(d^{O(d)} n)$ det.

$O(d^2 n + 2^{O(\sqrt{d} \log d)})$ rand.

ellipsoid method, interior-pt...

Badoiu-Har-Peled-Indyk's Alg'm (2002)

idea - coresets!

find small subset $S \subseteq P$ s.t.

$\text{min-rad}(S) \geq (1-\varepsilon)\, \text{min-rad}(P)$

Note - ε-kernels stronger, but have size $O\left(\frac{1}{\varepsilon^{(d-1)/2}}\right)$

exponential in d

idea - construct coreset $\{q_0, q_1, \dots\}$ iteratively

pick any $q_0 \in P$

for $i = 1, 2, \dots$

compute $B_i = \text{MEB}(\{q_0, \dots, q_{i-1}\})$

say center c_i, radius r_i

q_i = ~~some pt outside $(1+\varepsilon) B_i$~~

farthest pt of P from c_i

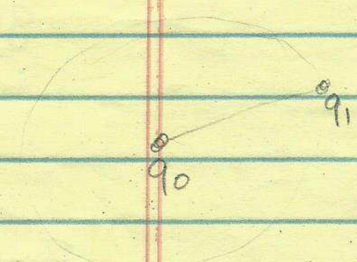
Original analysis: coreset of size $O\left(\frac{1}{\varepsilon^2}\right)$

indep. of d, amazingly!

Badiou-Clarkson's analysis (2003): size $O\left(\frac{1}{\varepsilon}\right)$.

optimal (lower bd $\Omega\left(\frac{1}{\varepsilon}\right)$).

Note:

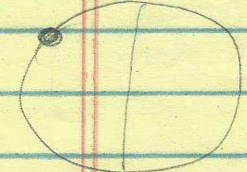


$\frac{r^*}{2} \leq r_2 \leq r_3 \leq r_4 \leq \cdots \leq r^*$

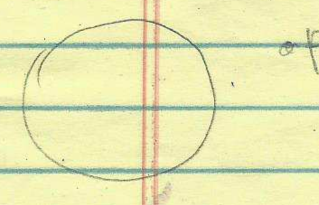
Will show r_i increases rapidly to r^*.

W.l.o.g. assume $r^* = 1$.

Facts a) every hemisphere of MEB(S) passes thru a pt of S

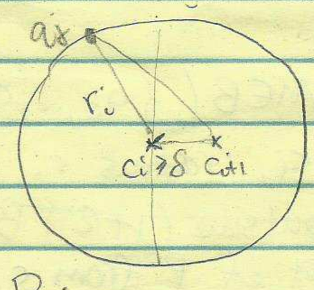


b) if p outside MEB(S), then MEB($S \cup \{p\}$) has p on its boundary.



Fix δ.

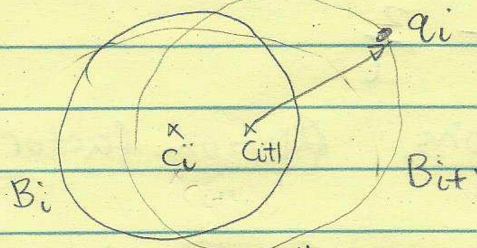
Case 1. $\|c_i - c_{i+1}\| \geq \delta$



By Fact (a), $\exists\, q_j \in \partial B_i$, $j < i$, s.t. $\angle q_j c_i c_{i+1} \geq 90^\circ$

$\Rightarrow r_{i+1} \geq \|c_{i+1} - q_j\| \geq \sqrt{r_i^2 + \delta^2}$

Case 2. $\|c_i - c_{i+1}\| < \delta$.



By Fact B,
$q_i \in \partial B_{i+1}$

$$\Rightarrow r_{i+1} = \|c_{i+1} - q_i\|$$
$$\geq \|c_i - q_i\| - \delta \geq 1 - \delta.$$

$$\therefore \; r_{i+1} \geq \min\left\{ \sqrt{r_i^2 + \delta^2},\ 1-\delta \right\}.$$

Choose δ s.t. $r_i^2 + \delta^2 = (1-\delta)^2 = 1 - 2\delta + \delta^2$

$$\delta = \frac{1 - r_i^2}{2}$$

$$\Rightarrow \boxed{r_{i+1} \geq \frac{1 + r_i^2}{2}}$$

$r_2 \geq \frac{1}{2}$, $r_3 \geq 0.625$, $r_4 \geq 0.695$, ...

How to solve recurrence:

let $r_i = 1 - y_i$

$$1 - y_{i+1} \geq \frac{1 + 1 - 2y_i + y_i^2}{2} \Rightarrow y_{i+1} \leq y_i - \frac{y_i^2}{2}$$

let $y_i = 1/z_i$

$$\frac{1}{z_{i+1}} \leq \frac{1}{z_i} - \frac{1}{2z_i^2} = \frac{1}{z_i}\left(1 - \frac{1}{2z_i}\right)$$

$$\Rightarrow z_{i+1} \geq \frac{z_i}{1 - \frac{1}{2z_i}} \geq z_i\left(1 + \frac{1}{2z_i}\right) = z_i + \frac{1}{2}$$

$\Rightarrow z_i \geq -i/2$

$\Rightarrow r_i \geq 1 - \frac{2}{i}$

With $i = \lceil 2/\varepsilon \rceil$ iterations, approx factor $1-\varepsilon$.

Rmk: Badoiu-Clarkson showed opt. coreset size $= \lceil 1/\varepsilon \rceil$.

runtime $\boxed{O\left(\frac{1}{\varepsilon} dn + \frac{1}{\varepsilon} * \text{cost of MEB alg'm on } O(\tfrac{1}{\varepsilon}) \text{ pts}\right)}$.

in $O(\frac{1}{\varepsilon})$ dims

[appl: diameter with $\sqrt{2}+\varepsilon$ approx factor]

Badoiu-Clarkson's Second Alg'm

idea - don't use coresets, but still iterative.

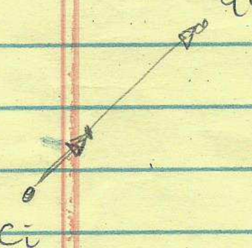
pick any $c_1 \in P$

for $i = 1, 2, \ldots$

q_i = farthest pt of P from c_i

$c_{i+1} = c_i + \frac{1}{i+1}(q_i - c_i)$

^ carefully chosen multiplier



related to gradient descent

Frank-Wolfe's alg'm [ML...]

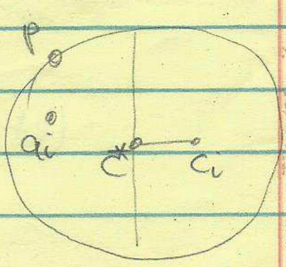
Analysis Sketch: Say $r^* = 1$.

Let c^* := opt center, B^* = opt ball

Let $t_i = \|c^* - c_i\|$.

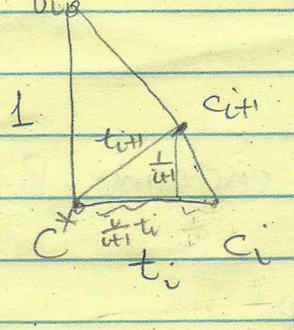
By Fact (a), $\exists\, p \in \partial B^*$ s.t.

$\angle p c^* c_i \geq 90°$



Then $\angle q_i c^* c_i \geq 90°$

"Worst" case $\angle q_i c^* c_i = 90^\circ$, $\|c^* - q_i\| = 1$.



$$t_{i+1}^2 = \left(\frac{i}{i+1} t_i\right)^2 + \left(\frac{1}{i+1}\right)^2$$

Let $z_i = (i t_i)^2$

$$\Rightarrow \quad z_{i+1} = z_i + 1 \quad \Rightarrow \quad z_i \le i$$

$$\Rightarrow \quad t_i = \frac{\sqrt{z_i}}{i} \le \frac{1}{\sqrt{i}}$$

With $i = \frac{1}{\varepsilon^2}$, approx factor $1 + \varepsilon$

runtime $\boxed{O\left(\frac{1}{\varepsilon^2} dn\right)}$

Rmk: combine two alg'ms

$\Rightarrow$ runtime $\boxed{O\left(\frac{1}{\varepsilon} dn + \frac{1}{\varepsilon^4} d\right)}$ or $\frac{1}{\varepsilon^5}$

[Allen-Zhu, Liao, Yuan '16: $\tilde{O}\left(\frac{1}{\sqrt{\varepsilon}} \sqrt{d}\, n + dn\right)$]

What if you are allowed only 1 pass ? with $O(d)$ space

<u>Streaming</u> alg'm

approx factor 2 is easy

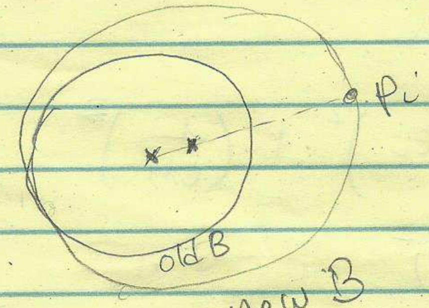
Zarrabi-Zadeh & C.'s Streaming Alg'm: (2006)

let $P = \{p_1, p_2, \ldots, p_n\}$

$B = \phi$

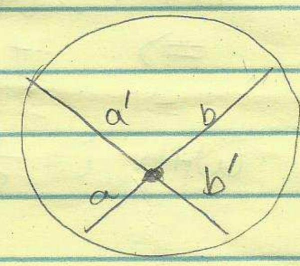
for $i = 1$ to n do

$B \leftarrow$ smallest ball enclosing B and p_i



approx factor 1.5

(pf not obvious)



Fact $ab = a'b'$

(high school geometry!)

Agarwal-Sharathkumar's Streaming Alg'm (SODA'10)

idea - maintain multiple balls $B_1, \ldots, B_u$

s.t. $P \subseteq \bigcup_{j=1}^{u} (1+\varepsilon) B_j$

$K_1 = \phi, \ B_1 = \phi, \ l = u = 0$

for $i = 1$ to n do

$\quad$ if $p_i \notin \bigcup_{j=l}^{u} (1+\varepsilon) B_j$ {

$\quad\quad u = u+1$

size $O(1/\varepsilon)$ by Badoiu-Clarkson $\rightarrow$ $K_u = \left(\frac{\varepsilon^2}{10}\right)$-coreset of $K_l \cup \ldots \cup K_{u-1} \cup \{p_i\}$

$\quad\quad B_u = MEB(S_u)$, say center c_u, radius r_u

$\quad\quad$ while $r_l < \frac{\varepsilon}{3} r_u$ do $l = l+1$.

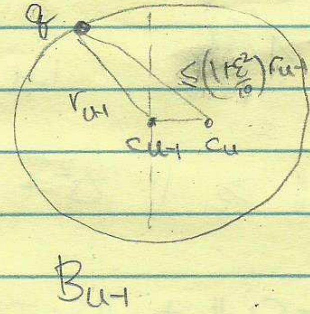
$\quad\quad\quad$ [$(1+\varepsilon) B_u$ covers $(1+\varepsilon) B_l$, so can drop B_l]

$\quad$ }

return $B = MEB\left(\bigcup_{j=l}^{u} K_j\right)$

Obs $\quad r_u \geq (1 + \Theta(\varepsilon^2))\, r_{u-1}$.

Pf: (similar to analysis of Badoiu et al.)



Case 1. $\|c_{u-1} - c_u\| \geq \frac{\varepsilon}{2} r_{u-1}$

By Fact (a), $\exists q \in K_{u-1}$ on ∂B_{u-1} s.t. $\angle q\, c_{u-1} c_u \geq 90°$

$$\Rightarrow (1+\tfrac{\varepsilon^2}{10})\, r_u \geq \sqrt{r_{u-1}^2 + \|c_{u-1} - c_u\|^2}$$
$$\geq \sqrt{1 + (\tfrac{\varepsilon}{2})^2}\; r_{u-1}$$
$$\geq \left(1 + \frac{\varepsilon^2}{8}\right) r_{u-1}$$

Case 2. $\|c_{u-1} - c_u\| < \frac{\varepsilon}{2} r_{u-1}$

$$\left(1 + \tfrac{\varepsilon^2}{10}\right) r_u \geq \|c_u - p_i\|$$
$$\geq \|c_{u-1} - p_i\| - \|c_{u-1} - c_u\|$$
$$\geq (1+\varepsilon)\, r_{u-1} - \frac{\varepsilon}{2} r_{u-1}$$
$$= (1 + \Theta(\varepsilon))\, r_{u-1}$$

$\square$

$$\Rightarrow \quad |u-\ell| \le O\left(\log_{(1+\Theta(\varepsilon^2))} \frac{1}{\varepsilon}\right) = O\left(\frac{\log \frac{1}{\varepsilon}}{\log(1+\Theta(\varepsilon^2))}\right)$$

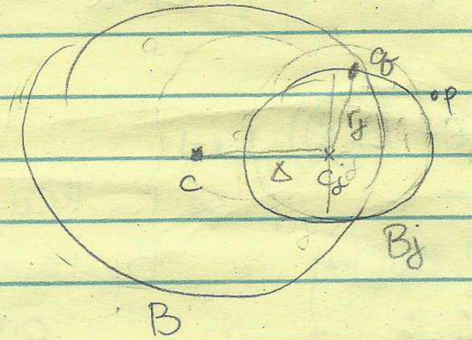
$$= O\left(\frac{1}{\varepsilon^2} \log \frac{1}{\varepsilon}\right)$$

$$\Rightarrow \quad \text{space } O\left(\frac{d}{\varepsilon^4} \log \frac{1}{\varepsilon}\right)$$

Approx factor:

Say B has center c, radius r

Let $p \in P$. Say $p \in (1+\varepsilon)B_j$



By Fact (a), $\exists\, q \in K_j$, on ∂B_j s.t. $\angle c c_j q \ge 90°$

let $\Delta = \|c - c_j\|$

$$\Rightarrow \quad r \ge \|c-q\| \ge \sqrt{\Delta^2 + r_j^2}$$

$$\begin{aligned} \|c-p\| &\le \|c-c_j\| + \|c_j - p\| \\ &\le \Delta + (1+\varepsilon) r_j \\ &\le (1+\varepsilon)\sqrt{2}\sqrt{\Delta^2 + r_j^2} \\ &\le (1+\varepsilon)\sqrt{2}\, r. \end{aligned}$$

$$\Rightarrow \quad P \subseteq \sqrt{2}(1+\varepsilon) B \quad \Rightarrow \quad r \le r^* \le \sqrt{2}(1+\varepsilon) r$$

$$\Rightarrow \quad \text{factor } \boxed{\sqrt{2}(1+\varepsilon)}$$

Rmk: improved analysis:

Agarwal-Sharathkumar	1.367	$((1+\sqrt{3})/2)$
C.-Pathak '11	1.22	(computer-assisted!)
lower bd	$>$ 1.207	

(other problems: min enclos. cylinder $< 5+\varepsilon$
2-center $1.8+\varepsilon$, etc.)