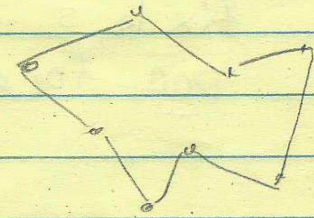
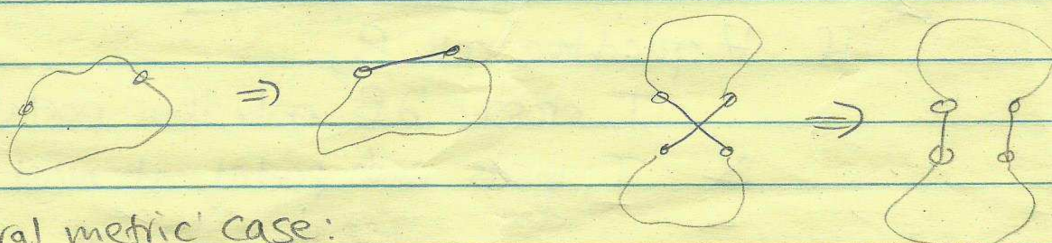
Euclidean Traveling Salesman Problem (TSP)

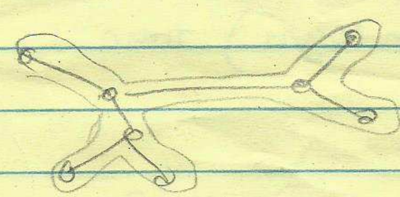
Given n pts $P \subset \mathbb{R}^2$,
find shortest tour C^* thru all pts



Note: opt tour is a polygon with
vertices at P & does not self-intersect



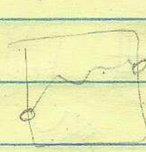
General metric case:
- 2-approx by MST



(3/2)-approx by Christofides '76
better than 3/2?? open

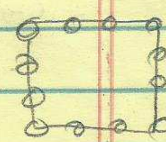
Geom. Case:
→ Arora '96 :
Mitchell '96 : } PTAS!

idea - shifted quadtree + DP



Say min bounding square has side length 1.
Round pts to grid of side length ε/n
$\Rightarrow$ total additive error $\leq \frac{\varepsilon}{n} \cdot n = \varepsilon \leq \varepsilon |C^*|$
Build quadtree (uncompressed ok)
$\Rightarrow$ depth $O(\log \frac{n}{\varepsilon}) = O(\log n)$; # nodes: $O(n \log n)$

Fix k.



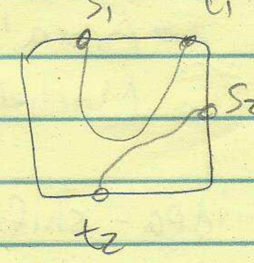
Def For each quadtree cell B,
place k evenly spaced pts on ∂B
called portals.

Def A tour T is portal-respecting
if $\forall$ quadtree cell B,
T crosses ∂B only thru portals
& T visits each portal at most 2 times.

Lemma can find exact shortest portal-respecting tour
in $O(2^{O(k)} n \log n)$ time.

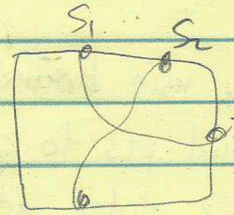
Pf: By D.P.

Subproblem Given quadtree cell B
& list of portal pairs $(s_1, t_1), \ldots, (s_b, t_b)$, call this an "interface" ($b \le 2k$)
find shortest set of b portal-respecting paths
from s_1 to $t_1, \ldots, s_b$ to t_b
thru all pts in $P \cap B$.

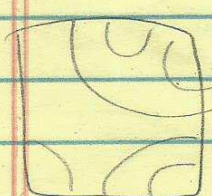


\# interfaces
$\le k^{O(k)}$

\# legal interfaces
$\approx$ strings of $\le 2k$ balanced parentheses
$\approx$ Catalan's number
$\le 2^{O(k)}$



not legal



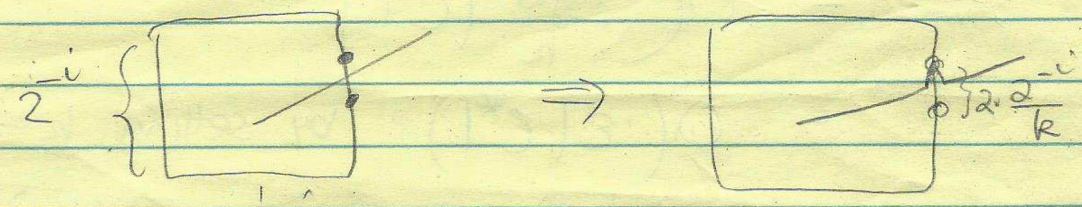
legal

~~()(())(()())~~

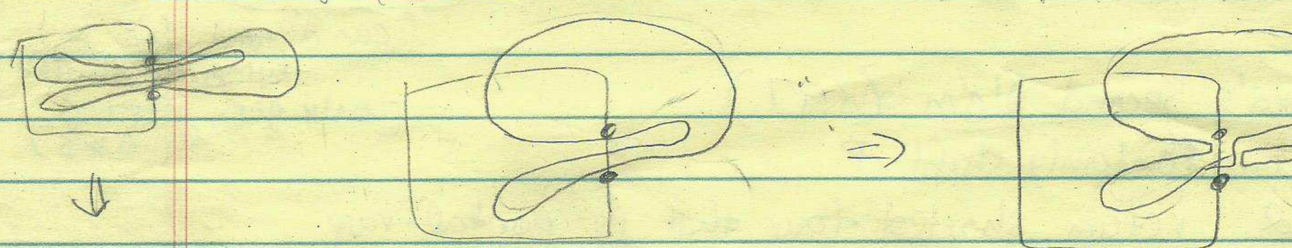
total # subproblems $\leq O\left(2^{O(k)} n \log n\right)$.
total time $O\left(\left(2^{O(k)}\right)^4 n \log n\right)$. $\square$

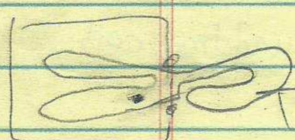
Arora's Alg'm:
1. randomly shift P
2. return shortest portal-resp tour.

Analysis:
Round C^* into portal-resp tour by adding detours

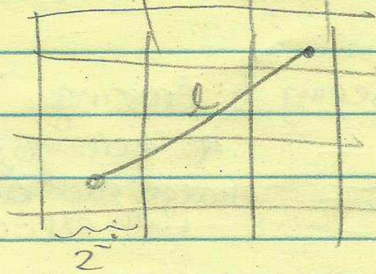


If portal is visited > 2 times, patch first.



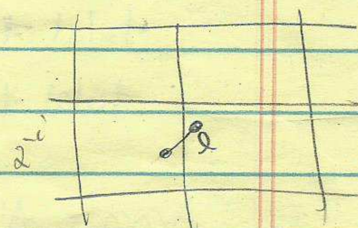


Total error?
Fix edge pq of C^*, of length ℓ.
Consider grid of side length 2^{-i}.
If $\ell \geq 2^{-i}$, # times pq crosses grid bdry $= O\left(\frac{\ell}{2^{-i}}\right)$



If $l < 2^{-i}$,

$$E[\#\text{ times } pq \text{ crosses grid bdry}] \leq \frac{l}{2^{-i}} + \frac{l}{2^{-i}} = O\left(\frac{l}{2^{-i}}\right)$$



$$\Rightarrow E[\#\text{ times } C^* \text{ crosses grid bdry}] = O\left(\frac{|C^*|}{2^{-i}}\right)$$

$$\Rightarrow E[\text{total error}] = O\left(\sum_{i=0}^{O(\log n)} \frac{|C^*|}{2^{-i}} \cdot \frac{2^{-i}}{k}\right)$$

$$= O\left(\frac{\log n}{k} |C^*|\right)$$

$$= O(\varepsilon |C^*|) \quad \text{by setting } k = \tfrac{1}{\varepsilon}\log n$$

Runtime: $2^{O(\frac{1}{\varepsilon}\log n)} n \log n = \boxed{n^{O(1/\varepsilon)}}$

[can derand by trying all shifts]

Arora's Second Alg'm ('97)

[only gives QPTAS for $d \geq 3$]

1. randomly shift P
2. return shortest tour that is portal-resp & crosses bdry of each quadtree cell $\leq b$ times

$\#$ interfaces $\leq k^{O(b)}$

$$\Rightarrow \boxed{O\left(n\left(\frac{\log n}{\varepsilon}\right)^{O(1/\varepsilon)}\right)} \quad \text{by setting } k \approx \tfrac{1}{\varepsilon}\log n, \; b \approx \tfrac{1}{\varepsilon}$$

by more careful analysis

(do patching when seeing b crossings)

bottom-up

$\#$ crossings decrease exponentially

charge each original crossing with exp.-decaying wt using spanners

Rmk - Rao-Smith '95 $O(2^{(1/\varepsilon)^{O(1)}} n \log n)$

- extends to any const $d \geq 3$

& other problems e.g. min Steiner tree

k-MST, ...