Euclidean Traveling Salesman Problem (TSP)

Given n pts \( P \subseteq \mathbb{R}^2 \), find shortest tour \( C^* \) thru all pts

Note: opt tour is a polygon with
vertices at \( P \) & does not self-intersect

General metric case:
2-approx by MST

\( (3/2) \)-approx by Christofides '76
better than 3/2 ? open

Geome Case:
\( \Rightarrow \) Arora '96: \( \exists \) PTAS!
Mitchell '96:

idea - shifted quadtree + DP

Say min bounding square has side length \( \varepsilon \).
Round pts to grid of side length \( \varepsilon / n \)
\( \Rightarrow \) total additive error \( \leq \varepsilon . n = \varepsilon \leq \varepsilon |C^*| \)
Build quadtree (uncompressed ok)
\( \Rightarrow \) depth \( O(\log \frac{\varepsilon}{\sqrt{n}}) = O(\log n) \);
# nodes: \( O(n \log n) \).
Fix $k$.

**Def** For each quadtree cell $B$, place $k$ evenly spaced pts on $\partial B$ called portals.

**Def** A tour $T$ is portal-respecting if $\forall$ quadtree cell $B$, $T$ crosses $\partial B$ only thru portals & $T$ visits each portal at most 2 times.

**Lemma** There exists an exact shortest portal-respecting tour in $O(2^{O(k)} n \log n)$ time.

**Pf:** By DP

**Subproblem** Given quadtree cell $B$ & list of portal pairs $(s_1, t_1), \ldots, (s_b, t_b)$, $b \leq 2k$, find shortest set of $b$ portal-respecting paths from $s_1$ to $t_1$, $\ldots$, $s_b$ to $t_b$ thru all pts in $\partial B$.

# interfaces $\leq k \cdot O(k)$

# legal interfaces $\sim$ strings of $k \cdot k$ balanced parentheses

$\sim$ Catalan's number

$\leq 2^{O(k)}$
total # subproblems \leq O\left( 2^{O(k)} n \log n \right)

total time \quad O\left( (2^{O(k)})^4 n \log n \right)

Arora's Alg:
1. randomly shift $P$
2. return shortest portal-resp tour

Analysis:
Round $C^*$ into portal-resp tour by adding detours

If portal is visited > 2 times, patch first

Fix edge $pq$ of $C^*$, of length $l$
Consider grid of side length $2^{-i}$
If $l \geq 2^{-i}$, # times $pq$ crosses grid boundary = $O\left( \frac{r}{2^i} \right)$
If $2^{-i}$, 

\[
E \left( \text{# times } p q \text{ crosses grid bdry} \right) \leq \frac{2^{-i}}{2^{-i}} + \frac{2^{-i}}{2^{-i}} = O \left( \frac{1}{2^{-i}} \right) \]

\[
E \left( \text{# times } C^* \text{ crosses grid bdry} \right) = O \left( \frac{1}{2^{-i}} \right)
\]

\[
E \left( \text{total error} \right) = O \left( \sum_{i=0}^{\infty} \frac{1}{2^{-i}} \right) = O \left( \log n \left( \frac{\log n}{1} \right) \right) = O \left( \frac{\log n}{1} \right)
\]

by setting $k = \frac{1}{\epsilon} \log n$

**Runtime:** $2^{O(\log n)} n \log n = \Omega \left( \frac{n \log n}{1} \right)$

(can be reduced by trying all shifts)

**Aurora's Second Alg. ('97)**

1. Randomly shift $P$

2. Return shortest tour that is partial-wrap & crosses bdry of each quadtree cell 8 times

\[
\# \text{ interfaces } \leq O(6)
\]

\[
\Rightarrow \left[ O \left( n \log n \right) \right] \text{ by setting } k = \frac{1}{\epsilon} \log n
\]

by more careful analysis

(do patching when seeing $b$ crossings)

(bottom-up)

(charge each original crossing with exp.-decaying wt)
Rmk - Rao-Smith '95 $O(2^{(1/3)} n \log n)$
- extends to any const $d \geq 3$
& other problems e.g. min Steiner tree $k$-MST, ...