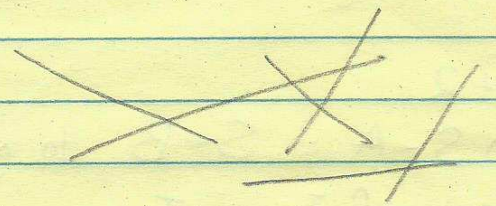
More Separator-Based Techniques

indep set for line segments, general curves, etc. ← "string graphs"



may have large union complexity! (& large cliques!)

Agarwal-Mustafa '04: $O(\sqrt{n}\log n)$-approx for line segs (or $\sqrt{OPT}$)
(by divide & conquer + Erdös-Szekeres + DP)

unweighted → Fox-Pach '11: $O(n^{\varepsilon})$-approx in polytime $n^{(1/\varepsilon)^{O(1/\varepsilon)}}$

weighted → Adamaszek-Wiese '13: $(1+\varepsilon)$-approx in quasi-polytime $n^{(\frac{1}{\varepsilon}\log n)^{O(1)}}$ (QPTAS)
(also works for rectangles)

(simplified/generalized by Har-Peled, ...)

Fox-Pach's Method

Fact 1 (Fox-Pach '08)

For intersection graph of n curves S in $\mathbb{R}^2$, with m intersections

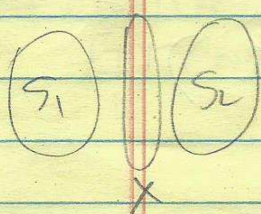
$\exists$ partition of S into S_1, S_2, X s.t.

$|S_1|, |S_2| \le 2n/3$

no intersections between S_1, S_2

$|X| \le O(\sqrt{m})$.

(if a curve has k intersection pts, add wt $1/k$ to each)



(i.e. sparse ⇒ ∃ good separators)

[Pf: apply wt'ed Lipton-Tarjan planar-graph separator thm with intersection pt as vertices]

(Rmk: can be strengthened with m = intersecting pairs where each pair intersects $O(1)$ times (Fox-Pach '10 / Matoušek '13 / Lee '17)

Fact 2 (Fox-Pach-Toth '11)

For intersection graph of n curves S in $\mathbb{R}^2$,

if intersections $m > \delta n^2$,

then $\exists$ disjoint subsets $A, B \subseteq S$ of size $\Omega(\delta n)$ s.t. every curve in A intersects every curve in B.

(i.e. dense ⇒ ∃ large balanced bi-clique)

Alg'm

Case 2: $m > \delta n^2$,

apply Fact 2

recurse in $S-A$, $S-B$ to get I_{S-A}, I_{S-B}

return $I =$ larger of I_{S-A}, I_{S-B}

Case 1: $m \le \delta n^2$,

apply Fact 1

recurse in $S_1, S_2; X$ to get I_{S_1}, I_{S_2}, I_X

return $I =$ larger of $I_{S_1} \cup I_{S_2}$, I_X

Approx factor: to show $|OPT(S)| \le f(n)\,|I|$.

Case 2: Know $OPT(S) \subseteq S-A$ or $\subseteq S-B$

$$\Rightarrow |OPT(S)| \le \max\{|OPT(S-A)|, |OPT(S-B)|\}$$
$$\le \max\{f((1-\delta)n)|I_{S-A}|, f((1-\delta)n)|I_{S-B}|$$
$$\le f((1-\delta)n)\,|I|.$$

Case 1. $$|OPT(S)| \le |OPT(S) \cap S_1| + |OPT(S) \cap S_2| + |OPT(S) \cap X|$$
$$\le f(2n/3)|I_{S_1}| + f(2n/3)|I_{S_2}| + f(\sqrt{\delta}n)|I_X|$$
$$\le (f(2n/3) + f(\sqrt{\delta}n))\,|I|.$$

Set $f(n) = f\left(\frac{2}{3}n\right) + f(\sqrt{\delta}\,n)$

solves to $O(n^{\varepsilon})$ for suff. small δ

[expand 1st term $\Rightarrow$ $f(n) = O\left(\log\frac{1}{\delta}\right) f(\sqrt{\delta}\,n)$

$\Rightarrow$ $f(n) = n^{O\left(\frac{\log \log \frac{1}{\delta}}{\log \frac{1}{\delta}}\right)}$

Set $\log\frac{1}{\delta} \sim \frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$, i.e. $\frac{1}{\delta} \sim \left(\frac{1}{\varepsilon}\right)^{O(1/\varepsilon)}$]

Runtime:

Case 1. $T(n) \le T(n_1) + T(n_2) + T(x) + \text{poly}(n)$

$n_1 + n_2 + x = n$.

Case 2. $T(n) \le 2\,T((1-\delta)n) + \text{poly}(n)$

$\Rightarrow T(n) = O\left(n^{\frac{\log 2}{\log 1/(1-\delta)}}\right) \le n^{O(1/\delta)}$

$= n^{(1/\varepsilon)^{O(1/\varepsilon)}}$

Adamaszek-Wiese's QPTAS

"Fact 1" Given n disjoint line segs Q,

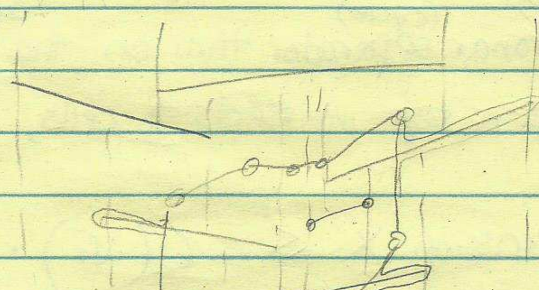
$\exists$ polygon X with $O(\sqrt{n})$ edges

s.t. # segs of T inside $X \le \frac{2}{3}n$

" " outside " $\le \frac{2}{3}n$

segs crossing X $= 0$.

[Pf: apply wt'ed planar-graph (cycle-) separator thm to the dual of vertical decomp. $VD(Q)$]



if a seg touches k cells, add wt $1/k$ to each

Standard idea - for exact algm: apply Fact 1 to (unknown) opt sol'n Q^*

guess X, then recurse

(# choices for X $\le n^{O(\sqrt{OPT})}$)

$T(OPT) \le n^{O(\sqrt{OPT})}\, 2 \cdot T(\frac{2}{3}OPT)$

$\Rightarrow T(OPT) \le n^{O(\sqrt{OPT})} \cdot n^{O(\sqrt{\frac{2}{3}OPT})} \cdot n^{O(\sqrt{\frac{4}{9}OPT})} \cdots \le n^{O(\sqrt{OPT})} \le n^{O(\sqrt{n})}$

(Cutting Lemma)

Fact 2 Given n disjoint line segs Q, & r,

cutting → $\exists$ decomp. into $O(r)$ cells

each intersecting $O(\frac{n}{r})$ segs.

[Rough Pf: pick sample R of size r
[Clarkson-Shor '88] return $VD(R)$]

New idea - combine!

Main Lemma Given n disjoint line segs Q, & r,

$\exists$ polygon X with $O(\sqrt{r})$ edges

s.t. # segs inside $X \leq \frac{2}{3}n$

" " outside " $\leq \frac{2}{3}n$

segs crossing $X \leq O(\frac{n}{\sqrt{r}})$

Pf: apply (wt'ed) planar (cycle) separator thm on the dual of cutting from Fact 2

(if a seg intersects k cells, add wt $1/k$ to each cell)

segs crossing $X \leq O(\sqrt{r}) \cdot O(\frac{n}{r})$. $\square$

Alg'm:

guess X, remove segs crossing X, recurse

Runtime:

$T(OPT) \leq n^{O(\sqrt{r})} \cdot 2\, T(\frac{2}{3} OPT)$

$\Rightarrow T(OPT) \leq n^{O(\sqrt{r} \log OPT)}$

Additive error:

$$E(OPT) \le \begin{cases} E(OPT_1) + E(OPT_2) + O\left(\frac{OPT}{\sqrt{r}}\right) & \text{if } OPT \ge \sqrt{r} \\ \quad \text{where } OPT_1 + OPT_2 \le OPT, & OPT_1, OPT_2 \le \frac{2}{3} OPT \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow E(OPT) = O\left(\frac{OPT}{\sqrt{r}} \log OPT\right)$$

Set $\sqrt{r} = \frac{1}{\varepsilon} \log OPT \Rightarrow$ factor $1+\varepsilon$ in $n^{O(\frac{1}{\varepsilon}\log^2 OPT)}$ time.

Rmk: extends to weighted, curves/polygons, ...

Mustafa-Raman-Ray '15: QPTAS for wt'ed set cover for pseudo-disks, ...

Chuzhoy-Ene '16: improves runtime to $n^{O(\frac{1}{\varepsilon}\log\log n)^{O(1)}}$ for rectangles

Hardness of Approx

How to rule out PTASs / QPTASs? reductions!

Fact indep set / vertex cover for graphs of deg 3 is APX-hard

$\Rightarrow$ any polytime alg'm has approx factor $\geq$ specific const > 1, assuming $P \neq NP$.

[Pf: uses PCP Thm ...]

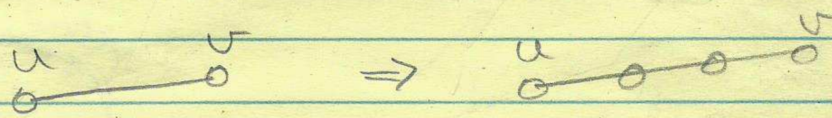
Corollary 1 (Chlebik, Chlebikova '05) indep set for boxes in 3D is APX-hard

Pf: by reduction from deg-3 IS

Given deg-3 graph $G = (V, E)$ with n vertices & $\frac{3n}{2}$ edges

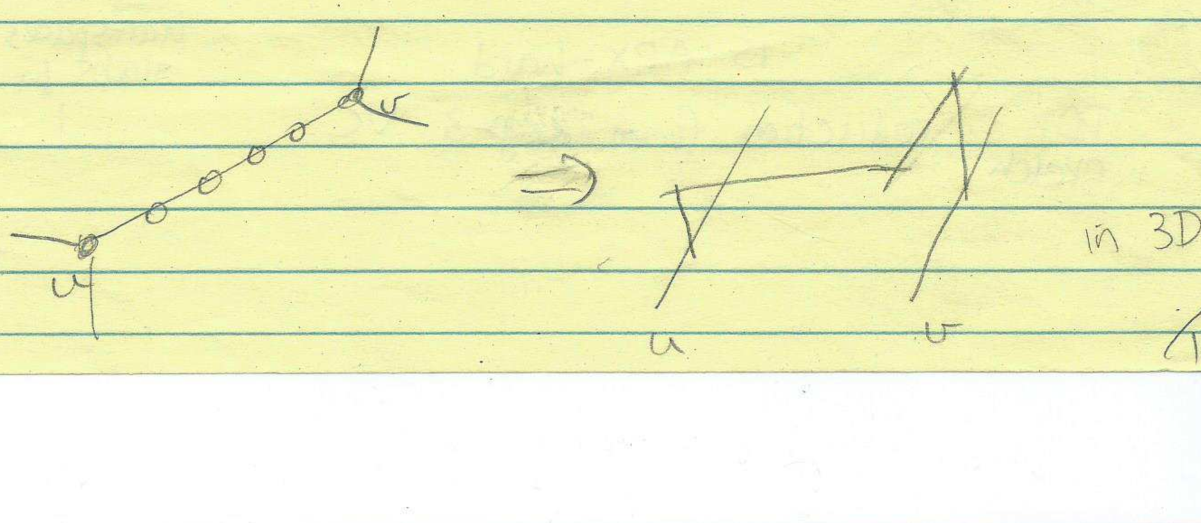
$\left(\frac{n}{4} \leq OPT \leq n\right)$

Obs if we subdivide an edge by adding 2 vertices, max IS size increases by exactly 1



Subdivide every edge by adding 4 vertices to get G'

$\left(OPT' = OPT + \frac{3n}{2} \cdot 2 \right)$



reduction preserves approx, up to consts ("L-reduction")

[if we get sol'n I' in G' with $|I'| \geq (1-\delta_0)\,OPT'$

then we get a sol'n I in G with

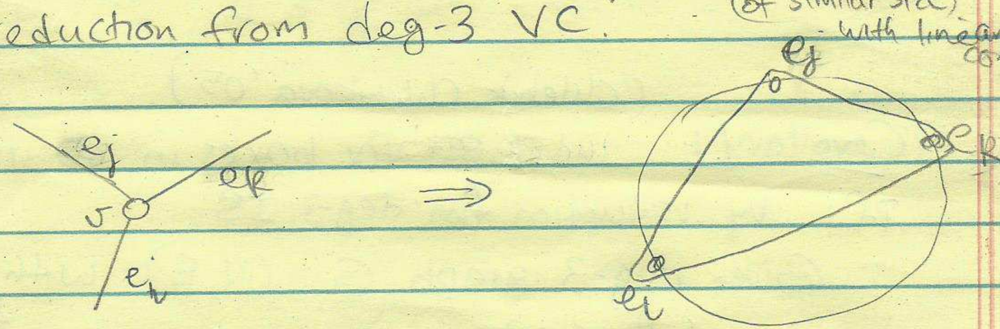
$$|I| = |I'| - 3n \geq (1-\delta_0)OPT' - 3n \geq (1-\delta_0)(OPT-3n)-3n$$

$$= (1-\delta_0)\,OPT - 3\delta_0 n \geq (1-13\delta_0)\,OPT.$$

$\square$

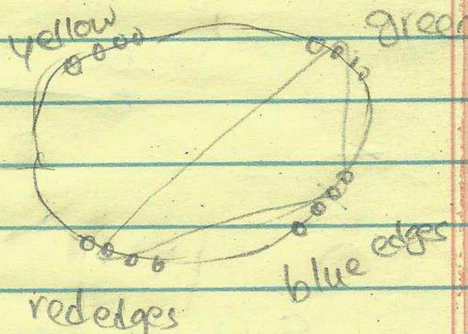
<u>Corollary 2</u> [Sariel '09] Set cover for fat triangles (of similar size) with linear union complexity is APX-hard!

Pf: reduction from deg-3 VC.



how to ensure fatness?

find edge-coloring with 4 colors



<u>Corollary 3</u> [C.-Grant '12] Set cover for fat rectangles (of similar size) in 2D

unit balls in 3D

halfspaces in 4D

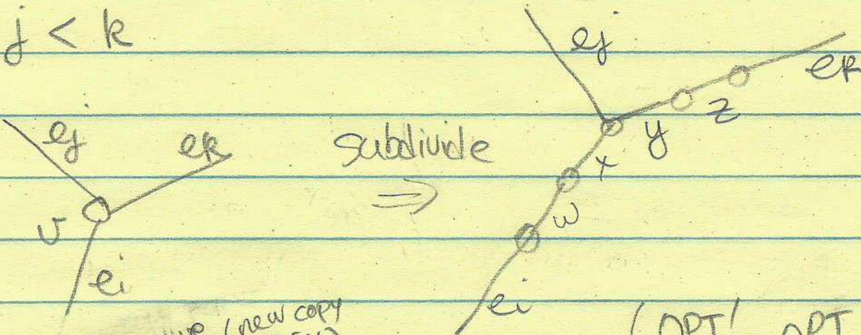
slabs in 2D

⋮

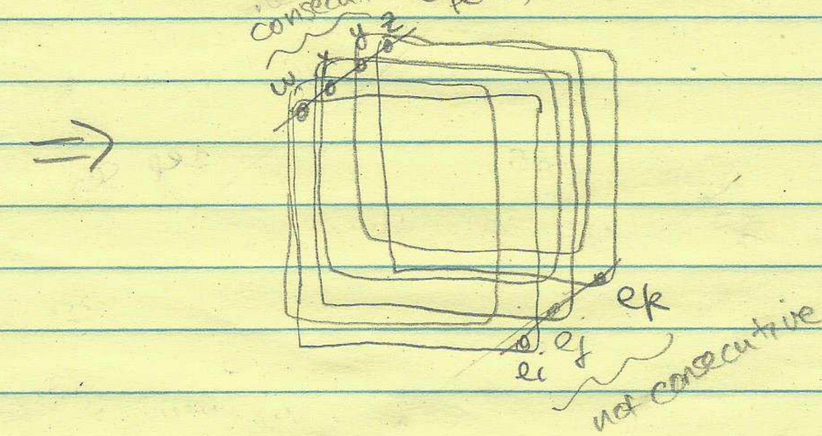
is APX-hard

Pf sketch: reduction from deg-3 VC.

Say $i < j < k$



$\left(OPT' = OPT + \frac{3n}{2} \cdot 2\right)$



(note: not pseudodisks)