LP Rounding (Cont'd)

weighted set cover for objs with low union complexity
e.g. disks in $\mathbb{R}^2$

Recall: Given n objs S,
$R \subseteq S$ is an ***ε-net*** if $\forall p$,
$\text{depth}_p(S) \geq \varepsilon n \Rightarrow p$ is covered by R.

Basic Thm Given n objs S of total wt W,
$\exists$ ε-net of weight $O\left(\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)\cdot\frac{W}{n}\right)$.

Pf: as shown, can take random sample with sampling prob. $\approx \frac{\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}}{n}$.

$\Rightarrow$ expected wt $= O\left(\sum_{i=1}^{n} w_i \frac{\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}}{n}\right)$. □

[Varadarajan'10 / CGKS'12]

Main Thm For disks in $\mathbb{R}^2$,
$\exists$ ε-net of weight $O\left(\frac{1}{\varepsilon}\cdot\frac{W}{n}\right)$.

[in rounding, $\hat{S}$ has size $\hat{n} \approx \sum_{i=1}^{n} Kx_i$
weight $\hat{W} \approx \sum_{i=1}^{n} K w_i x_i \leq K\,OPT$

$\frac{1}{\varepsilon} \approx \sum_{i=1}^{n} x_i$

$\Rightarrow O\left(\frac{1}{\varepsilon}\cdot\frac{\hat{W}}{\hat{n}}\right) = O(OPT)$

$\Rightarrow$ approx factor $O(1)$].

idea - relax depth of sample

Standard Def R is a p-sample of S if
each obj in S is put in R with prob p independently

New Def R is a quasi-p-sample of S if
$\forall s \in S$, $\Pr[s \in R] \le p$. (but not independent)

Goal: describe a quasi-$O(\frac{1/\varepsilon}{n})$-sample that is an ε-net [$O(\frac{1}{k})$]
$\Rightarrow$ expected wt $= O\left(\sum_{i=1}^{n} w_i \frac{1/\varepsilon}{n}\right) = O\left(\frac{1}{\varepsilon} \cdot \frac{W}{n}\right)$.
(don't need indep!)

Set $k = \varepsilon n$. [prev. constructions not work!]
[rand alg'm oblivious to wts!]

Idea - decrease k gradually

Lemma Let R be a $\left(\frac{1}{2} + c\sqrt{\frac{\log k}{k}}\right)$-sample of S.

Then for fixed p,
$$\text{depth}_p(S) \ge k \Rightarrow \text{depth}_p(R) \ge \frac{k}{2}$$
with prob $1 - \frac{1}{k^{\Theta(c)}}$.

Pf: by Chernoff
$$\mu = E[\text{depth}_p(R)] = \text{depth}_p(S) \cdot \left(\frac{1}{2} + c\sqrt{\frac{\log k}{k}}\right)$$
$$\ge \frac{k}{2} + c\sqrt{k \log k}$$
$$\Pr[\text{depth}_p(R) \ge \frac{k}{2}] \le \Pr[\text{depth}_p(R) - \mu \ge \underbrace{D}_{c\sqrt{k\log k}}]$$
$$\le \frac{1}{e^{\Theta(D^2/\mu)}} = \frac{1}{e^{\Theta(c \log k)}} = \frac{1}{k^{\Theta(c)}}$$
$\square$

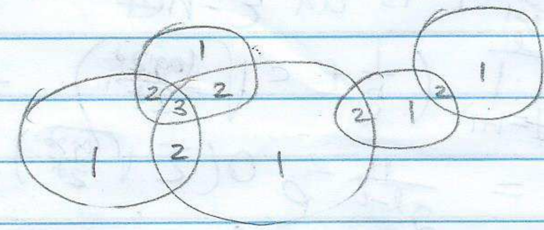
but we want it to work for <u>all</u> p?

<u>"Correction" Lemma</u> Let R be a $\left(\frac{1}{2} + c\sqrt{\frac{\log k}{k}}\right)$ - sample of S.
Can construct quasi- $\frac{1}{k^{\Theta(c)}}$ - sample A of S s.t.

$$\forall p, \ \text{depth}_p(S) \geq k \Rightarrow \text{depth}_p(R) \geq \frac{k}{2}$$

or p is covered by A.

Pf: Recall that $(\leq k)$-level has $O(nk)$ vertices & thus $O(nk)$ cells



each k-level cells is contained in $O(k)$ objs

$\Rightarrow \exists$ obj s containing $O\left(\frac{nk \cdot k}{n}\right) = O(k^2)$ k-level cells

remove s, recursively construct A

now put back s

for all p with $\text{depth}_p(S) \geq k$,

if $\text{depth}_p(S - \{s\}) \geq k$, ok by induction

otherwise, $\text{depth}_p(S) = k$ & p is in s.

diff. p's is $O(k^2)$ — if $\text{depth}_p(R) \geq k/2$ ok

otherwise, add s to A. — Prob $\leq \frac{1}{k^{\Theta(c)}}$ by Prev Lemma

$$\Pr[s \text{ is added to } A] \leq O(k^2) \cdot \frac{1}{k^{\Theta(c)}} = \frac{1}{k^{\Theta(c)}}$$

□

Pf of Main Thm:

Apply Correction Lemma $\ell = \log k$ times to get $R_\ell, \ldots, R_0, A_\ell, \ldots, A_1$

where R_{i-1} is $\left(\frac{1}{2} + c\sqrt{\frac{\log 2^i}{2^i}}\right)$-sample of R_i $(R_\ell = S)$.

A_i is quasi-$\frac{1}{(2^i)^{\Theta(c)}}$-sample of R_i

$\forall p$, $\text{depth}_p(R_i) \geq 2^i \implies \text{depth}_p(R_{i-1}) \gtrsim 2^{i-1}$

or p is covered by A_i

then $\text{depth}_p(S) \geq k \implies \text{depth}_p(R_0) \geq 1$

or p is covered by $\bigcup_{i=1}^{\ell} A_i$

$\Rightarrow N = R_0 \cup \bigcup_{i=1}^{\ell} A_i$ is an ε-net

R_i is $\prod_{j=i+1}^{\ell} \left(\frac{1}{2} + c\sqrt{\frac{\log 2^j}{2^j}}\right)$ - sample of S

$$= \frac{1}{2^{\ell-i}} e^{O\left(\sum \sqrt{\frac{\log 2^j}{2^j}}\right)} = O\left(\frac{1}{k/2^i}\right)$$

A_i is quasi-$O\left(\frac{1}{k/2^i} \cdot \frac{1}{(2^i)^{\Theta(c)}}\right)$ sample of S

$$= O\left(\frac{1}{k} \cdot \frac{1}{(2^i)^{\Theta(c)}}\right)$$

$\Rightarrow N$ is quasi-$O\left(\frac{1}{k} + \frac{1}{k}\sum_{i=1}^{\ell} \frac{1}{(2^i)^{\Theta(c)}}\right)$ - sample of S

$$= O\left(\frac{1}{k}\right).$$

"shallow cell complexity"

Rmk: if # k-level cells is $O(n\,\varphi(n)\,k^{O(1)})$

approx factor $O(\log \varphi(n))$