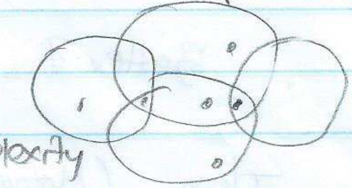
given n objs S
m pts P.



LP Rounding (Cont'd)

set cover for disks or pseudo-disks
or objs w. low union complexity

Warm-Up: case when all pts in P have large depth

Def Given n objs S, a subset $R \subseteq S$ is an ε-net if every pt with depth $> \varepsilon n$ in S is covered by R.

Rmk: ε-approximation $\Rightarrow$ ε-net $\left(\left| \frac{\text{depth}_S(p)}{n} - \frac{\text{depth}_R(p)}{|R|} \right| \leq \varepsilon \right)$

Thm $\exists$ ε-net of size $O(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon})$.

Pf: Pick random sample $R \subseteq S$ of size r

For fixed pt p with depth $> \varepsilon n$, in S,

$$\Pr[p \text{ not covered by } R] \leq \left(1 - \frac{r}{n}\right)^{\varepsilon n}$$

$$\leq \left(e^{-r/n}\right)^{\varepsilon n} = \frac{1}{e^{\varepsilon r}}$$

set $r = \text{const} \cdot \frac{1}{\varepsilon} \log n$

$$\leq \frac{1}{n^{100}}.$$

$$\Rightarrow \Pr[\text{fail}] \leq O\left(n^2 \cdot \frac{1}{n^{100}}\right) \ll 1.$$

$\Rightarrow$ $\exists$ ε-net of size $O(\frac{1}{\varepsilon} \log n)$

How to reduce $\log n$ to $\log \frac{1}{\varepsilon}$:

size $O(\frac{1}{\varepsilon^2} \log \frac{1}{\varepsilon})$

Fact If R' is an ε'-approximation of S & R is an ε-net of R', then R is an $(\varepsilon + \varepsilon')$-net of S.

$\square$

Rmk: works for any set system with bounded shattering dim

Better?

Thm (Matoušek-Seidel-Welzl '90)
For disks in $\mathbb{R}^2$, $\exists$ ε-nets of size $O(1/\varepsilon)$ $\qquad r = \frac{1}{2\varepsilon}$

one Pf: by shallow cutting lemma
$\exists$ $O(r)$ cells each intersected by $O(\frac{n}{r})$ disks
covering all pts of depth $\le \frac{n}{r}$
$\Rightarrow$ $\exists$ $O(\frac{1}{\varepsilon})$ cells each intersected by $\le c\varepsilon n$ disks
covering all pts of depth $\le 2\varepsilon n$.

For each cell,
take $\frac{1}{c}$-net of size $O(1)$
$\Rightarrow$ total size $O(\frac{1}{\varepsilon})$
covering all pts of depth $\in [\varepsilon n, 2\varepsilon n]$.

Overall size $O(\frac{1}{\varepsilon} + \frac{1}{2\varepsilon} + \frac{1}{4\varepsilon} + \dots) = O(\frac{1}{\varepsilon})$. $\square$

set cover can be formulated as ILP:

$$z_{LP} = \min \sum_{i=1}^{n} x_i$$

$$\text{s.t.} \quad \sum_{\substack{\text{obj } i \\ \text{containing } p}} x_i \ge 1 \quad \forall p \in P$$

$$\cancel{x_i \in \{0,1\}}$$

$$0 \le x_i \le 1$$

LP relaxation ("covering LP")

Know $z_{LP} \le OPT$.

How to round LP sol'n?

create a multiset $\hat{S}$ where

obj i gets $\lfloor K x_i \rfloor$ copies for suff. large K.

Then $|\hat{S}| = \sum_{i=1}^{n} \lfloor K x_i \rfloor \leq K \cdot OPT$

$\forall p \in P$, depth of p in $\hat{S}$

$= \sum_{\substack{\text{obj } i \\ \text{containing } p}} \lfloor K x_i \rfloor \geq K - n.$

Return ε-net of $\hat{S}$ with $\varepsilon = \frac{K-n}{K \cdot OPT} = \Omega\left(\frac{1}{OPT}\right)$

$\Rightarrow$ size $O\left(\frac{1}{\varepsilon}\right) = O(OPT)$.

$\Longrightarrow$ approx factor $O(1)$.

Rmks:

- for arb. objs with bounded shattering dim,

  approx factor $O(\log OPT)$

  ("better" than $O(\log n)$ for general set cover)

  if $\exists$ ε-net of size $O\left(\frac{1}{\varepsilon}\mu\left(\frac{1}{\varepsilon}\right)\right)$,

  approx factor $O(\mu(OPT))$

- if union complexity is $O(n\varphi(n))$, (e.g. fat triangles $\varphi(n) = O(\log \log n)$)

  $\exists$ ε-net of size $O\left(\frac{1}{\varepsilon}\varphi\left(\frac{1}{\varepsilon}\right)\right)$ [Clarkson-Varadarajan '05]

  improves to $O\left(\frac{1}{\varepsilon}\log\varphi\left(\frac{1}{\varepsilon}\right)\right)$ [Varadarajan '09/ Aronov-Ezra-Sharir '09]

  $\Longleftarrow$ $O(\log\varphi(OPT))$ approx factor

- for hitting set of rectangles, in $\mathbb{R}^2$,

  Aronov-Ezra-Sharir: $\exists$ ε-net of size $O\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right)$

  $\Rightarrow$ $O(\log\log OPT)$ approx factor

- weighted??

Alternative to LP: by "multiplicative weights update"
[Brönnimann-Goodrich '94]

Guess ε with $4\,OPT \le \frac{1}{\varepsilon} \le 8\,OPT$

create multiset $\hat{S}$ where initially each obj in S has multiplicity 1

repeat {

if some pt p has depth $< \varepsilon|\hat{S}|$ in $\hat{S}$ (multiplicity included)

for each obj i covering p

double multiplicity of obj i

else return a $(1/\varepsilon)$-net R of $\hat{S}$ of size $O(\frac{1}{\varepsilon}) = O(OPT)$

}

Analysis: At each iteration,

$|\hat{S}|$ increases by factor $1+\varepsilon$

$\Rightarrow$ after ℓ iterations, $|\hat{S}| \le (1+\varepsilon)^{\ell}\, n$

Let T^* be opt sol'n. $(|T^*| = OPT)$

In each iteration, some obj in T^* has its multiplicity doubled

$\Rightarrow$ after ℓ iterations, some obj in T^* has been doubled $\ge \frac{\ell}{OPT}$ times

$$|\hat{S}| \ge 2^{\ell/OPT}$$

$$\Rightarrow 2^{\frac{\ell}{OPT}} \le (1+\varepsilon)^{\ell} n \le e^{\varepsilon \ell} n \le 2^{2\varepsilon\ell} n \le 2^{\frac{\ell}{2OPT}} n$$

$$\Rightarrow 2^{\frac{\ell}{2OPT}} \le n \Rightarrow \ell \le O(OPT \log n)$$

$\Rightarrow$ alg'm must terminate. $\square$