LP Rounding (Cont'd)

set cover for disks or pseudo-disks
or obj w. low union complexity

Warm-Up: case when all pts have large depth

Def: Given n objs $S$, a subset $R \subseteq S$ is an $\epsilon$-net
if every pt with depth $> \epsilon n$ in $S$
is covered by $R$.

Rank: $\epsilon$-approximation $\Rightarrow$ $\epsilon$-net

\[ \frac{\text{depth}(P) - \text{depth}(R)}{\epsilon n} \leq 1 \]

Thm: $\exists$ $\epsilon$-net of size $O\left(\frac{1}{\epsilon} \log \frac{1}{\epsilon} \right)$.

Pf: Pick random sample $R \subseteq S$ of size $r$.

For fixed pt $p$ with depth $> \epsilon n$ in $S$,

\[ \Pr[ p \text{ not covered by } R ] \leq \left( 1 - \frac{\epsilon}{n} \right)^{\epsilon n} \leq \left( e^{-\frac{\epsilon}{n}} \right)^{\epsilon n} = \frac{1}{e^{\epsilon}} \]

Set $r = \text{const} \cdot \log n$.

\[ \Rightarrow \Pr[\text{fail}] \leq O\left( \frac{n^2}{n^{100}} \right) \ll 1 \]

$\exists$ $\epsilon$-net of size $O\left( \frac{1}{\epsilon} \log n \right)$.

How to reduce $\log n$ to $\log \frac{1}{\epsilon}$:

Fact: If $R'$ is an $\epsilon$-approximation of $S$
& $R$ is an $\epsilon$-net of $R'$,
then $R$ is an $(\epsilon + \epsilon')$-net of $S$.

Rmk: works for any set system with bounded shattering dim.
Better?

Thin (Matoušek-Seidel-Welzl '90)

For disks in $\mathbb{R}^2$, $\exists$ $\varepsilon$-nets of size $O(\frac{1}{\varepsilon})$

one $P_i$; by shallow cutting lemma

$\exists$ $O(1)$ cells each intersected by $O(\frac{1}{\varepsilon})$ disks

covering all pts of depth $\leq \frac{1}{4}$

$\Rightarrow$ $\exists$ $O(\frac{1}{\varepsilon})$ cells each intersected by $\leq \varepsilon$ disks

covering all pts of depth $\leq 2 \varepsilon n$

For each cell,

take $\frac{1}{2}$-net of size $O(1)$

$\Rightarrow$ total size $O(\frac{1}{\varepsilon})$

covering all pts of depth $\varepsilon \in [\varepsilon n, 2 \varepsilon n]$

Overall size $O(\frac{1}{\varepsilon} + \frac{1}{2\varepsilon} + \frac{1}{4\varepsilon} + \ldots) = O(\frac{1}{\varepsilon})$. ⊙

Set cover can be formulated as ILP:

$$z_{ILP} = \min \sum_{i=1}^{n} x_i$$

s.t. $\sum_{i \in P} x_i \geq 1$, $\forall P \in P$

$\forall i \in \text{obj i containing } P$

$x_i \in \{0, 1\}$

LP relaxation

$0 \leq x_i \leq 1$ ("covering LP")

Know $z_{ILP} \leq \text{OPT}$. 
How to round LP sol'n?

Create a multiset \( \hat{S} \) where

- obj \( i \) gets \( \left\lceil \frac{K x_i}{L} \right\rceil \) copies for suff large \( K \).

Then \( |\hat{S}| = \sum \left\lceil \frac{K x_i}{L} \right\rceil \leq K \cdot \text{OPT} \)

4 p \in \hat{S}, depth of p in \( \hat{S} \)

\[ \sum_{\text{obj } i \text{ containing } p} L K x_i \geq K \cdot n. \]

Return \( \varepsilon \)-net of \( \hat{S} \) with \( \varepsilon = \frac{K \cdot n}{K \cdot \text{OPT}} = \mathcal{O}(\frac{1}{\text{OPT}}) \)

\[ \Rightarrow \text{size } O\left(\frac{1}{\varepsilon}\right) = O\left(\frac{1}{\text{OPT}}\right). \]

\[ \Rightarrow \text{approx factor } O(1). \]

Remarks:

- for arb. objs with bounded shattering dim,
  approx factor \( O(\log \text{OPT}) \)
  ("better" than \( O(\log n) \) for general set cover)
  if \( \exists \ \varepsilon \)-net of size \( O\left(\frac{1}{\varepsilon} \mu(\frac{1}{\varepsilon})\right) \),
  approx factor \( O(\mu(\text{OPT})) \)
  if union complexity is \( O(n \varepsilon \mu(\text{OPT})) \)
  \( \exists \ \varepsilon \)-net of size \( O\left(\frac{1}{\varepsilon} \log (\varepsilon)\right) \) \[ \text{[Clarkson - Varadarajan '05]} \]
  \[ \text{improves to } O\left(\frac{1}{\varepsilon} (\log \log (\varepsilon))\right) \text{ [Varadarajan '05]} \]

- for hitting set of rectangles in \( \mathbb{R}^2 \),
  Aronov-Ezra-Sharir: \( \exists \ \varepsilon \)-net of size \( O\left(\frac{1}{\varepsilon} \log (\varepsilon)\right) \)
  \[ \Rightarrow O\left(\log \log \text{OPT}\right) \text{ approx factor} \]

- weighted ??
Alternative to LP: by "multiplicative weights update"

[Bromberg-Goodrich '94]

Guess \( \varepsilon \) with \( 4\text{OPT} \leq \varepsilon \leq 8\text{OPT} \)

create multiset \( \hat{S} \) where initially each obj in \( S \) has multiplicity 1
repeat \( i \)

if some pt \( p \) has depth \( \leq \varepsilon |\hat{S}| \) in \( \hat{S} \) (multiplicities included)
	for each obj \( i \) covering \( p \)
	double multiplicity of obj \( i \)
else return a \((\varepsilon)\)-net \( R \) of \( \hat{S} \) of size \( \Omega(\varepsilon) = O(\text{OPT}) \)

Analysis: At each iteration,
\[ |\hat{S}| \text{ increases by factor } 1 + \varepsilon \]

\( \Rightarrow \) after \( i \) iterations,
\[ |\hat{S}| \leq (1 + \varepsilon)^i \]

Let \( T^* \) be opt sol'n. \((|T^*| = \text{OPT})\)

In each iteration, some obj in \( T^* \) has its multiplicity doubled

\( \Rightarrow \) after \( i \) iterations, some obj in \( T^* \) has been doubled

\[ |\hat{S}| \geq 2^{\varepsilon/\text{OPT}} \]

\( \Rightarrow 2^{\varepsilon/\text{OPT}} \leq (1 + \varepsilon)^{n} \leq e^{\varepsilon n} \leq 2^{\varepsilon n} \leq 2^{2^{\varepsilon n}} \leq 2^{2^{\varepsilon n}} \]

\( \Rightarrow \) \( e \leq 0(\text{OPT} \log n) \)

\( \Rightarrow \) alg'm must terminate.