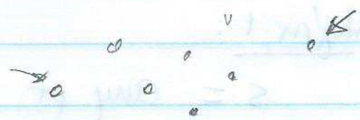
The Diameter Problem (farthest pair)

Given set P of n pts in $\mathbb{R}^d$,

find $\Delta^* = \max_{p,q \in P} \|p-q\|$



Euclidean dist.

Known exact alg'ms:

brute force $O(dn^2)$ time

$d=2$ $O(n \log n)$

$d=3$ $O(n \log n)$ randomized by Clarkson-Shor '88

Complicated { deterministic by Ramos '01

$d=4$ $\tilde{O}(n^{4/3})$ Matoušek '92

⋮

$\tilde{O}(n^{2 - \frac{2}{\lceil d/2 \rceil + 1}})$

Relax problem:

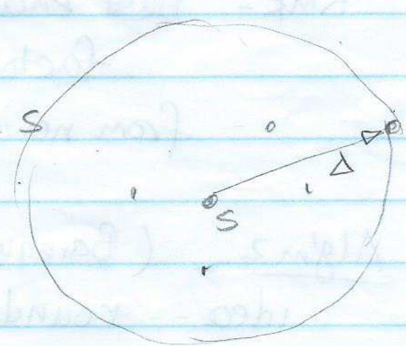
find Δ s.t. $\Delta \le \Delta^* \le c\Delta$

approx factor

Approx Alg'm 0:

s = any pt of P

return Δ = farthest dist from s



Runtime: $O(dn)$

Approx factor:

P lies in ball $B(s, \Delta)$

$\Rightarrow \Delta^* \le 2\Delta \Rightarrow$ factor $\boxed{2}$

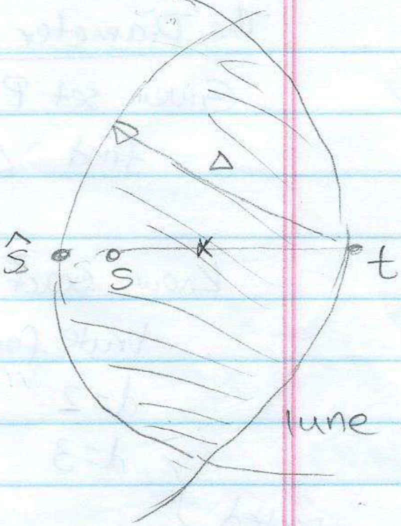
(works in any metric space)

Alg'm 1:

$s =$ any pt

$t =$ farthest neighbor of s

return $\Delta =$ farthest dist. from t



Runtime: $O(dn)$

Approx factor:

Let $\hat{s}$ be on $\overleftrightarrow{st}$ at dist Δ from t

$\forall p \in P, \quad \|p-t\| \le \Delta$

$$\|p-\hat{s}\| \le \|p-s\| + \|s-\hat{s}\|$$
$$\le \|t-s\| + \|s-\hat{s}\| = \Delta.$$

$\Rightarrow$ P lies in lune $B(\hat{s},\Delta) \cap B(t,\Delta)$

$\Rightarrow$ P lies in ball of radius $\frac{\sqrt{3}}{2}\Delta \Longrightarrow \Delta^* \le \sqrt{3}\Delta$

$\Rightarrow$ factor $\boxed{\sqrt{3}} \sim 1.733$ (works in any dim)

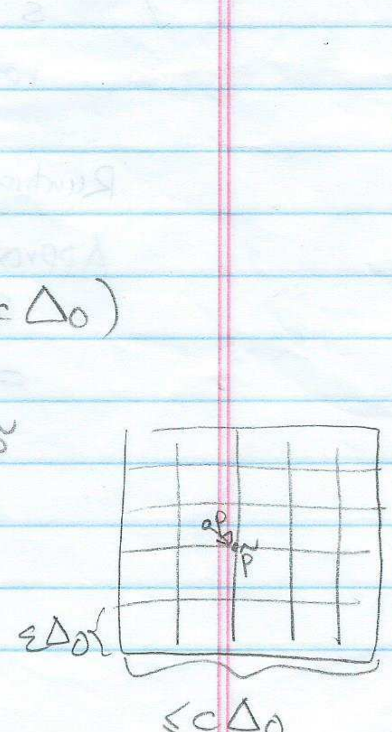
Rmk - best known $O(dn)$ alg'm has factor $\sim \sqrt{2}$

- from now on, assume d is const.

Alg'm 2 (Barequet-Har-Peled '99)

idea - rounding pts / grid

1. $\Delta_0 =$ c-approx. $(\Delta_0 \le \Delta^* \le c\Delta_0)$
2. form uniform grid of side length $\varepsilon\Delta_0$
3. round each $p \in P$ to nearest grid pt $\tilde{p}$
4. return $\Delta =$ diameter of $\{\tilde{p} : p \in P\}$



Runtime: # grid pts $= O\left(\left(\frac{c\Delta_0}{\varepsilon\Delta_0}\right)^d\right)$

$= O\left(\frac{1}{\varepsilon^d}\right)$

line 4 $O\left(\frac{1}{\varepsilon^{2d}}\right)$ by brute force

remove duplicates by hashing

total $\boxed{O\left(n + \frac{1}{\varepsilon^{2d}}\right)}$

Approx factor: Say $\Delta^* = \|p^* - q^*\|$ $(p^*, q^* \in P)$.

$\forall p \in P, \ \|p - \tilde{p}\| \le \frac{\varepsilon}{2}\sqrt{d}\,\Delta_0$

$\Rightarrow \ \big|\, \|p^* - q^*\| - \|\tilde{p}^* - \tilde{q}^*\| \,\big| \le \varepsilon\sqrt{d}\,\Delta_0$

$\Rightarrow \ |\Delta - \Delta^*| \le \varepsilon\sqrt{d}\,\Delta_0$

$\Rightarrow \ (1 - \varepsilon\sqrt{d})\Delta^* \le \Delta \le (1 + \varepsilon\sqrt{d})\,\Delta^*$

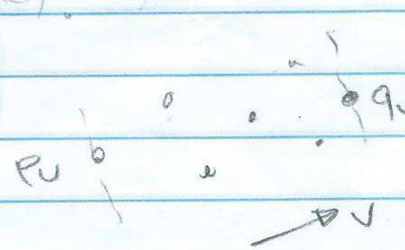
$\Rightarrow$ factor $1 + O(\varepsilon)$

readjust $\varepsilon \Rightarrow$ factor $1+\varepsilon$

Rmk - refinement: for each vertical grid line, keep only min & max pt

$\Rightarrow O\left(n + \frac{1}{\varepsilon^{2(d-1)}}\right)$

can we further improve ε-dependency?



Algm 3 (Agarwal-Matoušek-Suri '92)

idea - round directions

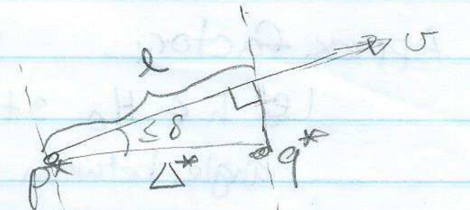
1. form a set V_δ of directions
2. for each $v \in V_\delta$, find extreme pts $p_v, q_v \in P$ (minimize $p_v \cdot v$, maximize $q_v \cdot v$)
3. return $\Delta = \max_{v \in V_\delta} \|p_v - q_v\|$.

Approx factor:

Say $\Delta^* = \|p^* - q^*\|$

let $v \in V_0$ s.t.

$\angle(p^* - q^*, v) \le \delta.$



$\Delta \geq \|p_v - q_v\|$

$\geq l \geq \Delta^* \cos\delta$

$\cos\delta \sim 1 - \frac{\delta^2}{2}$

$\Rightarrow$ factor $\frac{1}{\cos\delta} \approx 1 + \frac{\delta^2}{2}$

Set $\delta = \Theta(\sqrt{\varepsilon}) \Rightarrow$ factor $1+\varepsilon$

Runtime $\boxed{O\left(\frac{1}{\varepsilon^{(d-1)/2}}\, n\right)}$

Alg'm 4

idea - combine!

run Alg'm 2, but in last step, use Alg'm 3

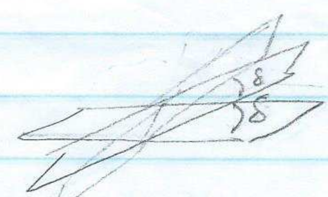
Approx factor $(1+\varepsilon)(1+\varepsilon) = 1 + O(\varepsilon)$

Runtime $O\left(n + \frac{1}{\varepsilon^{(d-1)/2}} \cdot \frac{1}{\varepsilon^{d-1}}\right) = \boxed{O\left(n + \frac{1}{\varepsilon^{3\left(\frac{d-1}{2}\right)}}\right)}$

Alg'm 5 (C.'01)

idea - better combination

- recurse in dimension



0. round pts as in Alg'm 2
1. form a set H_δ of hyperplanes thru z-axis of angle δ
2. for each $h \in H_\delta$
3. find diameter of projection to h
4. return max

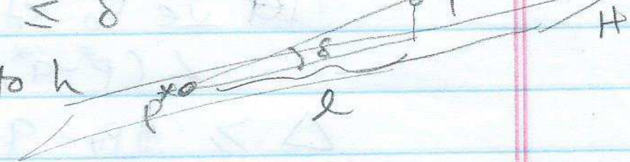
Approx. factor: Say $\Delta^* = \|p^* - q^*\|$.

Let $h \in H_0$ s.t.

angle between p^*q^* and $h \le \delta$

Then projection of p^*, q^* to h

has distance



$$\ell \geq \Delta^* \cos\delta \geq \frac{\Delta^*}{1+\varepsilon}$$ for $\delta = \Theta(\sqrt{\varepsilon})$ as before

Overall factor $(1+\varepsilon)^d = 1 + O(\varepsilon)$

Runtime:

$$T_d(n) = O\left(n + \frac{1}{\sqrt{\varepsilon}} T_{d-1}\left(\frac{1}{\varepsilon^{d-1}}\right)\right)$$

$$\Rightarrow T_d(n) = O\left(n + \frac{1}{\sqrt{\varepsilon}}\left(\frac{1}{\varepsilon^{d-1}} + \frac{1}{\sqrt{\varepsilon}} T_{d-2}\left(\frac{1}{\varepsilon^{d-2}}\right)\right)\right)$$

$$= O\left(n + \frac{1}{\varepsilon^{d-(1/2)}} + \frac{1}{\varepsilon} T_{d-2}\left(\frac{1}{\varepsilon^{d-2}}\right)\right)$$

$$= O\left(n + \frac{1}{\varepsilon^{d-(1/2)}} + \frac{1}{\varepsilon}\left(\frac{1}{\varepsilon^{d-2}} + \frac{1}{\sqrt{\varepsilon}} T_{d-3}\left(\frac{1}{\varepsilon^{d-3}}\right)\right)\right)$$

$$\vdots$$

$$= \boxed{O\left(n + \frac{1}{\varepsilon^{d-(1/2)}}\right)}$$