## Technique 5: LP Rounding

weighted indep set for objs with low union complexity

e.g. pseudo-disks

[C.-Har-Peled '09]

**Warm-Up:** case of small depth

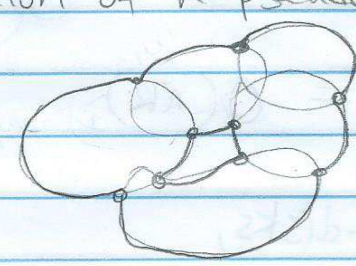
**Main Thm** Given $n$ weighted pseudodisks in $\mathbb{R}^2$, of total weight $W$,

if max depth $\le k$,

can find indep set of size $\Omega\left(\frac{n}{k}\right)$ (or weight $\Omega\left(\frac{W}{k}\right)$)

(not true in general)

**Fact 1** union of $n$ pseudo-disks has $O(n)$ vertices.



**Fact 2** $(\le k)$-level of $n$ pseudo-disks $S$ has $O(nk)$ vertices

\# vertices of depth $\le k$

**Pf 1:** by shallow cutting Lemma (Matoušek) $(r = n/k)$

$\exists$ $O(n/k)$ cells each intersected by $O(k)$ objs covering $(\le k)$-level

$\Rightarrow$ in each cell, $O(k^2)$ vertices

$\Rightarrow$ $O\left(\frac{n}{k} \cdot k^2\right)$ total. $\square$

**Pf 2:** [Clarkson-Shor '88] Pick rand. sample $R \subseteq S$ of size $\frac{n}{k}$

(each obj chosen w. prob $\frac{1}{k}$ indep)

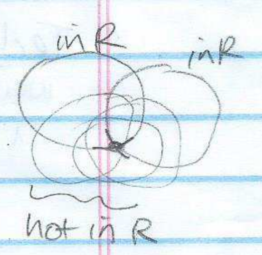
Let $V_k(S)$ = vertices of depth $\le k$ in $S$.

Let $V_0(R)$ = vertices of the union of $R$.

Fix $v \in V_k(S)$.

$$\Pr\left[ v \in V_0(R) \right] = \frac{1}{k} \cdot \frac{1}{k} \cdot \underbrace{\left(1 - \frac{1}{k}\right)^k}_{e^{-1}}$$

$$= \Omega\left(\frac{1}{k^2}\right)$$




$$\Rightarrow \quad E\left[ |V_0(R)| \right] \geq \Omega\left( \frac{1}{k^2} \cdot |V_k(S)| \right)$$

$$\Rightarrow \quad |V_k(S)| \leq O(k^2) \cdot \underbrace{E\left[ |V_0(R)| \right]}_{O\left(\frac{n}{k}\right) \text{ by Fact 1}}$$

$$= O(nk). \quad \square$$

Cor For $n$ pseudo-disks,
if max depth $\leq k$, intersection graph has $O(nk)$ edges.
In particular, $\exists$ obj with degree $O(k)$.

Fact 3 For a graph with $n$ vertices,
if every subgraph has a vertex of deg $\leq \Delta$, (called degeneracy)
then $\exists$ indep set of size $\geq \frac{n}{\Delta+1}$.
(or weight $\geq \frac{W}{\Delta+1}$)

Pf. (unweighted): by greedy
repeat {
pick vertex $v$ of deg $\leq \Delta$
remove $v$ & its neighbors
}

$n = \#$ vertices removed $\leq (\Delta+1) \cdot \#$ iterations

$\Rightarrow \#$ iterations $\geq \frac{n}{\Delta+1}$. $\square$

Pf (weighted): color the graph with $\Delta+1$ colors

pick color class with largest weight

How to color? by greedy

pick vertex $v$ of deg $\leq \Delta$

remove $v$, & recursively color

put back $v$, with color in $\{1, \ldots, \Delta+1\}$ diff from neighbors' colors $\square$

Pf of Main Thm: by Facts 2+3. $\square$

Back to LP rounding technique:

indep set can be formulated as integer linear program: (ILP)

"Wrong" formulation

$$\max \sum_{i=1}^{n} w_i x_i$$

$$\text{s.t. } x_i + x_j \leq 1 \quad \forall \text{ objs } i,j \text{ that intersect}$$

$$x_i \in \{0,1\}$$

better formulation

$$z_{LP} = \max \sum_{i=1}^{n} w_i x_i$$

$$\text{s.t. } \forall \text{ point } p, \sum_{\substack{\text{obj } i \\ \text{containing } p}} x_i \leq 1$$

$$\cancel{x_i \in \{0,1\}}$$

$$0 \leq x_i \leq 1$$

(linear programming)

LP relaxation can be solved in polytime ("packing LP")

by known LP alg'ms

Know $z_{LP} \geq$ OPT.

[Note: for unweighted, LP dual of the piercing problem]

How to "round" LP sol'n into integer sol'n? [Simplification of C.-Har-Peled]

Create a multiset $\hat{S}$ where
obj $i$ gets $\lceil K x_i \rceil$ copies for suff. large $K$.

Then $\hat{S}$ has total weight

$$\sum_{i=1}^{n} w_i \lceil K x_i \rceil \geq K z_{LP} \geq K \cdot OPT.$$

$\forall p$, depth of $p$ in $\hat{S}$

$$= \sum_{\substack{\text{obj } i \\ \text{containing } p}} \lceil K x_i \rceil \leq K + n$$

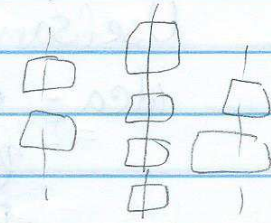
By Main Thm,
can find indep set of weight $\geq \Omega\left(\frac{K \cdot OPT}{K+n}\right) = \Omega(OPT)$

$\Rightarrow$ approx factor $\boxed{O(1)}$

Rmk: if union complexity is $O(n\varphi(n))$,
approx factor is $O(\varphi(n))$.

## LP Rounding (cont'd)

**Indep set of rectangles:**



divide & conquer: approx factor $\log_2 n$

combined w. DP: " " $\log_b n$ in $n^{O(b)}$ time

for weighted → C.-Har-Peled '09: factor $O\left(\frac{\log n}{\log\log n}\right)$ in polytime

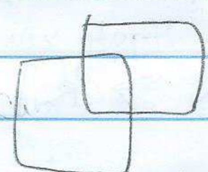
LP rounding

for unweighted → Chalermsook-Chuzhoy '09: factor $O(\log\log n)$ in polytime

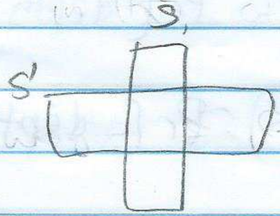
Adamaszek-Wiese '13: factor $1+\varepsilon$ in quasi-polytime $n^{\left(\frac{1}{\varepsilon}\log n\right)^c}$

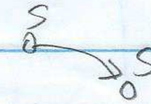
Chuzhoy-Ene '16: factor $1+\varepsilon$ in time $n^{\left(\frac{1}{\varepsilon}\log\log n\right)^c}$

2 types of intersections



type 1

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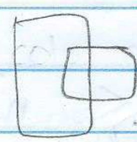


**Fact** For $n$ weighted rectangles without type 1 intersections, if max depth $\le k$, can find indep set of size $\Omega\left(\frac{W}{k}\right)$.

Pf: color with $k$ colors by defining directed intersection graph & give all sinks common color, remove, & repeat. $\square$

Use same LP relaxation     Know $z_{LP} \geq$ OPT.

idea – randomized rounding

put obj $i$ in $R$ w. prob. $\frac{x_i}{10}$.

put $i$ in $R'$ if $i \in R$ and all objs $j$ with $w_j > w_i$ & type-1 int. with obj $i$ are not in $R$.

return indep set $R''$ of $R'$ by Fact

$$\forall p,\ E[\text{depth of } p \text{ in } R] = \sum_{i=1}^{n} \frac{x_i}{10} \leq \frac{1}{10}$$

By Chernoff, $\mu \leq \frac{1}{10}$

$$\Pr[\text{depth of } p \text{ in } R > (1+\delta)\mu] \leq \left(\frac{1}{1+\delta}\right)^{(1+\delta)\mu}$$

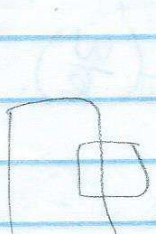
$$\Rightarrow \Pr[\text{depth of } p \text{ in } R > k] \leq \left(\frac{1}{k}\right)^{k/10}$$

$$\leq \frac{1}{n^{100}}$$ by setting $k = \text{const} * \frac{\log n}{\log\log n}$

$$\Rightarrow \Pr[\text{max depth} \geq k] \leq n^2 \cdot \frac{1}{n^{100}} \ll 1.$$

$\Rightarrow$ weight of $R'' \geq \Omega(\text{OPT}/k)$ $\Rightarrow$ approx factor $\boxed{O\left(\frac{\log n}{\log\log n}\right)}$

$$E[\text{weight of } R'] = \sum_{i=1}^{n} w_i \frac{x_i}{10}\Bigg(1 - \sum_{\substack{w_j > w_i \\ (i,j) \text{ type-1 int}}} \frac{x_j}{10}\Bigg)$$



$$\geq \frac{1}{10} z_{LP} - \frac{1}{100} \sum_{\substack{w_j > w_i \\ (i,j) \text{ type-1 int.}}} w_i x_i x_j$$

$$\geq \frac{1}{10} z_{LP} - \frac{1}{100} \sum_i w_i x_i \Bigg(\sum_{\substack{j \text{ contains} \\ \text{corner of } i}} x_j\Bigg) - \frac{1}{100} \sum_j w_j \Bigg(\sum_{\substack{i \text{ contains} \\ \text{corner of } j}} x_i\Bigg)$$

$$\geq \frac{1}{10} z_{LP} - \frac{4}{100} z_{LP} - \frac{4}{100} z_{LP} = \Omega(\text{OPT}).$$