Let $W(S)$ be the total depth of $S$ in $S$.

Fact 1. Union of $n$ pseudodisks has $O(n^2)$ vertices.

Fact 2. By shattering cutting lemma (Morton, 1982) $f(n) = \frac{c}{n^2}$.

Fact 3. $O(n^2)$ cells each intersected by $O(k)$ disks.

Combiner $C(k)$ - Level of $k$ vertices.

$C(k)$ = $\sum_{n=0}^{k} \binom{k}{n}$

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If max depth $\leq k$,

Think of given n pseudodisks in R^2 of total weight w & union with 1/pseudodisks.

Warm-up: case of small depth.

C. Hershberger, 09.

& g. pseudodisks

$(x, y) \in \min$.

Rounding $\mathcal{E}$.
Let $V_0(R) =$ vertices of the union of $R$.

Fix $v \in V_k(S)$.

$$\Pr \left[ v \in V_0(R) \right] = \frac{1}{k} \cdot \frac{1}{k^r} \left( 1 - \frac{1}{k} \right)^{k - 1}$$

$$= \Omega \left( \frac{1}{k^2} \right).$$

$$\Rightarrow E \left[ |V_0(R)| \right] \geq \Omega \left( \frac{1}{k^2} \cdot |V_k(S)| \right).$$

$$\Rightarrow |V_k(S)| \leq O(k^2) \cdot E \left[ |V_0(R)| \right]$$

$$= O(nk), \quad \Box$$

**Cor** For $n$ pseudo-disks, if max depth $\leq k$, intersection graph has $O(nk)$ edges.

In particular, 3 obj with degree $O(k)$.

**Fact 3** For a graph with $n$ vertices, if every subgraph has a vertex of deg $\leq \Delta$, then 3 indep set of size $\geq \frac{n}{\Delta + 1}$.

**(unweighted):** by greedy

repeat C

pick vertex $v$ of deg $\leq \Delta$

remove $v$ & its neighbors
n = \# vertices removed \leq (\Delta + 1) \cdot \# iterations

\Rightarrow \# iterations \geq \frac{n}{\Delta + 1}.

**Proof (weighted):** color the graph with \Delta + 1 colors
pick color class with largest weight
How to color? by greedy
pick vertex \( v \) of deg \( \leq \Delta \)
remove \( v \), \& recursively color
put back \( v \), with color in \([1, \ldots, \Delta + 1]\) diff from neighbors' colors

**Proof of Main Thm:** by Facts 2 + 3.

Back to LP rounding technique:

Indep set can be formulated as integer linear program:

\[
\text{max } \sum_{i=1}^{n} w_i x_i
\]

"Wrong" formulation

\[
\text{st. } x_i + x_j \leq 1 \quad \forall \text{obj } i, j \text{ that intersect } \sum_{i=1}^{n} x_i \leq 1
\]

Better formulation

\[
\text{st. } \forall \text{ point } p, \sum_{i=1}^{n} x_i \leq 1 \quad \text{obj } i \text{ containing } p
\]

x_i \in \{0, 1\}

0 \leq x_i \leq 1

LP relaxation can be solved in polytime ("rounding LP") by known LP algms
Know $\frac{2}{\gamma} \geq \text{OPT}.$

[Note: for unweighted, LP dual of the piercing problem]

How to "round" LP sol'n into integer sol'n? [Simplification of C.-Har-Peled]

Create a multiset $\tilde{S}$ where:
- obj $i$ gets $\lceil Kx_i \rceil$ copies for suff large $K$

Then $\tilde{S}$ has total weight
\[ \sum_{i} \omega_i \lceil Kx_i \rceil \geq K \mu_{\text{LP}} \geq K \cdot \text{OPT}. \]

\[ \forall p, \text{ depth of } p \text{ in } \tilde{S} \]
\[ = \sum_{\text{obj containing } p} \lceil Kx_i \rceil \leq K + n. \]

By Main Thm, can findrepid set of weight $\geq \Omega(\frac{K \cdot \text{OPT}}{K + n}) = \Omega(\text{OPT})$

\[ \Rightarrow \text{approx factor } O(1) \]

Rule: if union complexity is $O(n \cdot \gamma(n))$,
approx factor is $O(\gamma(n)).$
(LP Rounding (cont'd))

Indep set of rectangles

divide & conquer: approx factor \( \log n \)

Combined w. DP: \( \log^2 n \)
in \( n \cdot \text{polylog} n \) time

C. - Har-Peled '09: factor \( O(\frac{\log n}{\log \log n}) \) in polytime

Chalermsook-Chuzhoy '09: factor \( O(\log \log n) \) in polytime

Adamaszek-Wiese '13: factor \( 1 + \varepsilon \) in quasi-polytime \( \widetilde{O}(\frac{n}{(\varepsilon \log n)} \)

Chuzhoy-Ene '16: factor \( 1 + \varepsilon \) in time \( n \cdot \text{polylog} n \)

2 types of intersections

\[ \square \quad \square' \]

\[ \square \quad \square' \]

\[ \text{type 1} \quad \text{type 2} \]

Fact: For \( n \) rectangles without type 1 intersections, if \( \text{max depth} \leq R \), can find indep set of size \( \Omega(\frac{n}{R}) \).

Pf: Color with \( R \) colors

by defining directed intersection graph & give all sinks common color, remove, & repeat.
Use same LP relaxation - know $\frac{1}{10} Z_{LP} \geq OPT$.

Idea - randomized rounding

Put obj $i$ in $R$ w/ prob. $\frac{x_i}{10}$.

Put $i$ in $R'$ if $i \in R$ and all $obj j$ with $w_j > w_i$ are not in $R$.

Return indep set of $R'$ by Fact.

$U_p, E[\text{depth of } p \text{ in } R] = \sum_{i=1}^{n} \frac{x_i}{10} \leq \frac{\mu}{10}$

By Chernoff, $Pr[\text{depth of } p \text{ in } R \geq (1+\delta) \mu] \leq \left(\frac{1}{1+\delta}\right)^{\mu}$.

$\Rightarrow Pr[\text{depth of } p \text{ in } R \geq k] \leq \left(\frac{1}{k}\right)^{\frac{\mu}{10}} \leq \frac{1}{n^{100}}$ by setting $\delta = \frac{1}{k}$.

$\Rightarrow Pr[\text{max depth} \geq k] \leq \frac{n^2}{100n^{100}} \ll 1$.

$\Rightarrow \text{weight of } R' = 2OPT/k \Rightarrow \text{approx factor}$

$E[\text{weight of } R'] = \sum_{i=1}^{n} w_i x_i (1 - \sum_{j \in \text{i}} x_j)$

$\Rightarrow \frac{1}{10} Z_{LP} - \frac{1}{100} \sum_{i} w_i x_i x_j$

$\Rightarrow \frac{1}{10} Z_{LP} - \frac{1}{100} Z_{LP} - \frac{1}{100} Z_{LP} = \delta(2OPT)$. 

$\Rightarrow \frac{1}{100} Z_{LP} = 2OPT$. 

$\Rightarrow \frac{1}{10} Z_{LP} - \frac{1}{100} Z_{LP} = 2OPT - \frac{1}{10} Z_{LP}$. 

$\Rightarrow \frac{1}{100} Z_{LP} = \frac{1}{10} Z_{LP} - 2OPT$. 