$$\Rightarrow E(m) = O\left(\frac{m}{b}\cdot\sqrt{b}\right) = O\left(\frac{m}{\sqrt{b}}\right)$$

$\Rightarrow$ approx factor $1+O(\frac{1}{\sqrt{b}})$ set $b = 1/\varepsilon^2$

$\Rightarrow$ factor $1+\varepsilon$ in $n^{O(1/\varepsilon^2)}$ time

Rmk: extends to piercing

Technique 4: Local Search

Indep set for fat objs

Fix b.

$T = \phi$ (or any feasible sol'n)

repeat {

for each subset $D \subseteq T$ of size $\le b$

& $I \subseteq S - T$ of size $|D|+1$ do

if $T - D \cup I$ is a feasible sol'n

$T \leftarrow T - D \cup I$

} until stuck

return T

Very simple "Meta"-Alg'm

Runtime: # iterations $O(n)$

time per iteration $O(n^b \cdot n^{b+1} \cdot n) = n^{O(b)}$

Analysis: (C.-Har-Peled '09)

Multi-cluster version of Smith-Wormald's Separator Thm

Given intersection graph of n const-depth fat objects in $\mathbb{R}^2$, can partition into $V_1, \ldots, V_{O(n/b)}, X$

$\partial V_i \subseteq X$

$|V_i \cup \partial V_i| \le b$,

$|X| \le O(\frac{n}{\sqrt{b}})$, $\sum_i |\partial V_i| \le O(\frac{n}{\sqrt{b}})$

all objs intersecting S_i, excluding S_i

Let T^* be opt indep set, T be locally opt indep set

Apply thm to $T^* \cup T$ ← depth 2.

Then $|T^* \cap V_i| \leq |T \cap (V_i \cup \partial V_i)|$.

(else can delete $T \cap (V_i \cup \partial V_i)$ from T & insert $T^* \cap V_i$ to get larger indep set)

$$\Rightarrow \sum_i |T^* \cap V_i| \leq \sum |T \cap V_i| + \sum_i |\partial V_i|.$$

$$\Rightarrow |T^*| = \sum_i |T^* \cap V_i| + |T^* \cap X|$$

$$\leq \sum_i |T \cap V_i| + \sum_i |\partial V_i| + |T^* \cap X|$$

$$\leq |T| + O\left(\frac{|T|+|T^*|}{\sqrt{b}}\right)$$

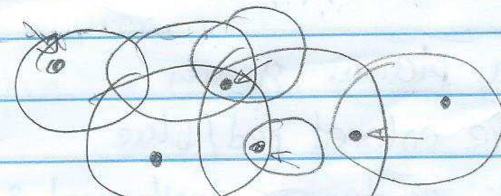
$$\Rightarrow |T^*| \leq \left(1 + O\left(\frac{1}{\sqrt{b}}\right)\right)|T|$$

RMK: works also for pseudodisks

$\Rightarrow$ factor $1 + O\left(\frac{1}{\sqrt{b}}\right)$ set $b = 1/\varepsilon^2$

$\Rightarrow$ factor $1+\varepsilon$ in $n^{O(1/\varepsilon^2)}$ time

Hitting set for disks in 2D: (discrete version)



Given n disks S, m pts P.
find min subset $Q \subseteq P$ that hits all of S.

similar local search alg'm (delete b, insert $b-1$)
runtime $n^{O(b)}$

Analysis: (Mustafa-Ray '09)

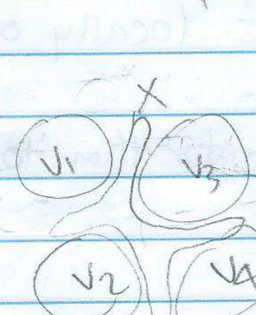
Multi-Cluster version of Planar Graph Separator Thm: Given planar graph $G=(V,E)$, $|V|=n$
can partition V into $V_1, \ldots, V_{O(n/b)}, X$ s.t.

$\partial V_i \subseteq X$

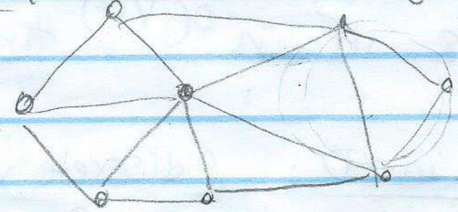
all neighbors of V_i excluding V_i

$|V_i \cup \partial V_i| \leq b$

$|X| \leq O\left(\frac{n}{\sqrt{b}}\right)$, $\sum_i |\partial V_i| \leq O\left(\frac{n}{\sqrt{b}}\right)$

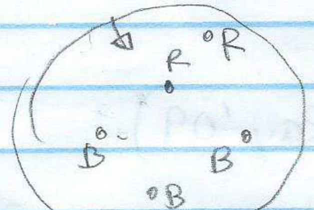


Def Given point set Q,
Delaunay triangulation (DT) is a graph over Q where
$q_i q_j$ is an edge iff $\exists$ circle thru $q_i q_j$ with no pts of Q inside
($q_i q_j q_k$ is a triangle iff circle thru $q_i q_j q_k$ has no pts of Q inside)



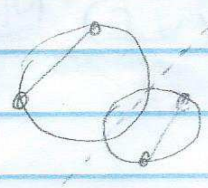
Facts (i) DT is a planar graph
(ii) if pts are colored red/blue,
every disk containing both red & blue pts must contain a bichromatic edge of DT.

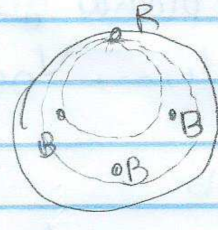
Pf of (ii):



shrink until all pts inside have same color

Pf of (i)





shrink while keeping contact with bdry pt until no pts inside $\square$

Let Q^* = opt hitting set, Q = locally opt hitting set.
Let G be DT of $Q^* \cup Q$
Apply multi-cluster planar graph separator thm to G.

Then $|Q \cap V_i| \leq |Q^* \cap (V_i \cup \partial V_i)|$

[else can delete $Q \cap V_i$ from Q

& insert $Q^* \cap (V_i \cup \partial V_i)$

to get smaller hitting set

color $Q \cap V_i$ red & all else blue

why? if disk s is hit by some pt in $Q \cap V_i$,

then s contains a bichromatic edge qq'

$q \in Q \cap V_i$, $q' \notin Q \cap V_i$

$\Rightarrow q' \in Q - V_i$ or $q' \in Q^* \cap (V_i \cap \partial V_i)$

$\Rightarrow$ s is still hit]

$$\Rightarrow \sum_i |Q \cap V_i| \leq \sum_i |Q^* \cap V_i| + \sum_i |\partial V_i|$$

$\vdots$

$$\Rightarrow |Q| \leq |Q^*| + O\left(\frac{|Q| + |Q^*|}{\sqrt{b}}\right)$$

$\Rightarrow$ factor $1 + O\left(\frac{1}{\sqrt{b}}\right)$ Set $b = 1/\varepsilon^2$

$\Rightarrow$ factor $1+\varepsilon$ in $n^{O(1/\varepsilon)}$ time