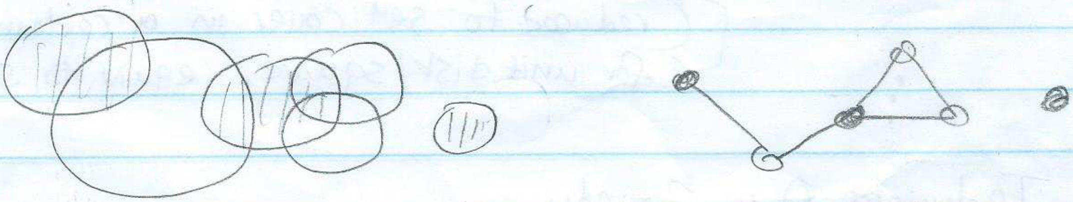
Geometric NP-Hard Problems

Independent Set

Given n objects, S, find max subset $T^* \subseteq S$
s.t. no 2 objects in T^* intersect



intersection graph

(objects: unit disks/squares/balls, arb. disks/squares, rectangles, line segs, triangles, polygons, ...)

(unweighted vs. weighted)

(Koebe's thm: planar graphs are contact graphs of disks)

[general case: $\approx n^{1-\varepsilon}$ approx]

Hitting Set / "Piercing" (Continuous Version)

Given n objects S, find min # pts X^*
that stab all objects

(Obs: piercing # $\geq$ independence #)

Hitting Set (Discrete Version)

Given n objects S, m pts P,
find min subset $X^* \subseteq P$ that stabs all of S

(unweighted vs. weighted)

Set Cover (Discrete Version)

[general case - $\sim \ln n$ approx.]

Given n pts P, m objects S
find min subset $T^* \subseteq S$ that covers all of P.
(equiv to hitting set in dual range space)
(for unit disk, equiv to hitting set)

Set Cover (Cont. Version)

Given n pts P,
find min # objects from a class e.g. unit disks that cover all of P.

Dominating Set

Given n objects S, find min $T^* \subseteq S$ s.t.
every object in $S - T^*$ intersects some object in T^*

(reduces to set cover in a certain range space)
(for unit disk/square, equiv to set cover)

⋮

Technique 0 : Greedy

indep set

$T = \emptyset$

repeat {

pick smallest object $t \in S$ (in radius)

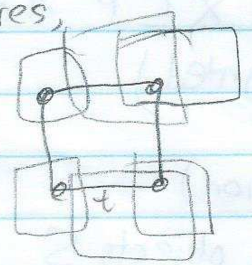
polytime

insert t to T, remove t & all objects intersecting t from S. (S_t)

}

Analysis:

For squares,



every square intersecting t bigger than t is stabbed by a corner of t

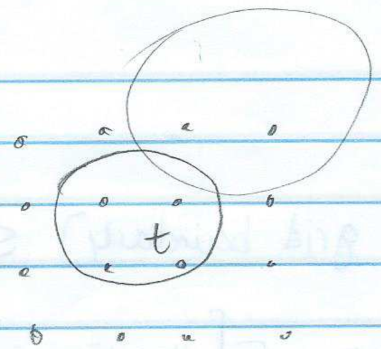
Let X^* be min piercing set

$\Rightarrow \quad |X^*| \leq 4|T|$

Recall $|T^*| \leq |X^*|$

$\Rightarrow \quad |T^*| \geq 4|T| \Rightarrow$ factor 4.

For disks,

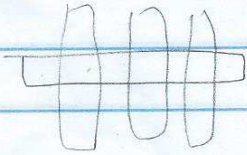


every disk intersecting t
bigger than t
can be stabbed by ≤ 16 pts

$\Rightarrow \quad |T^*| \geq 16\,|T| \quad \Rightarrow$ factor ≤ 16

For "fat objects", const factor

Rmks: - not work for arbitrary rectangles

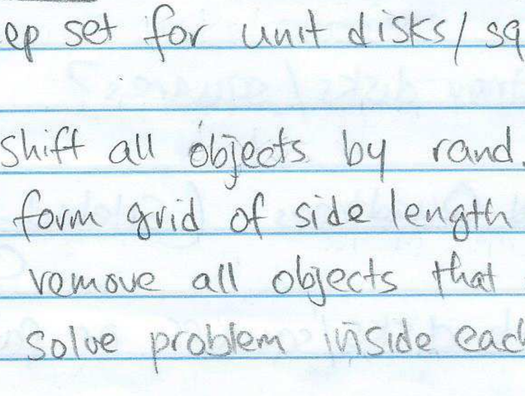


- works for piercing
(e.g. for squares, $|X| = 4|T| \leq 4|T^*| \leq 4|X^*|$)
- not work for weighted arb. squares

Technique 1: Shifted Grid (Hochbaum-Maass '85)
indep set for unit disks/squares or fat objects of "similar" size
(max r / min r = O(1))

1. shift all objects by rand. vector (in $[0, U]^2$)
2. form grid of side length b
3. remove all objects that cross grid cell boundaries
4. solve problem inside each grid cell by brute force

Sol'n has $\leq O(b^2)$ objects



Runtime:
$$O\Big(\sum_i n_i^{O(b^2)}\Big)$$
$$= n^{O(b^2)}.$$

Analysis:

Fix $t \in T^*$.

Prob[t crosses grid boundary] $\leq \frac{2}{b}$.

$\Rightarrow E[|T|] \geq E[$ # objs in T^* not cross grid boundary $]$

$\geq (1 - \frac{2}{b}) |T^*|$

Set $b = \Theta(\frac{1}{\varepsilon}) \Rightarrow (1 \pm \varepsilon)$-approx in time $n^{O(1/\varepsilon^2)}$

<u>polytime</u> <u>approx.</u> <u>scheme</u> (PTAS)

Rmk: - can be derandomized, works for weighted

- works for piercing for fat objs of similar size

— not work for cont. vers. of hitting set

What about arbitrary disks / squares?

<u>Technique 2</u>: Shifted Quadtrees [Erlebach-Jansen-Seidel '01 / C.'01]

indep set for arb. disks / squares, or fat objects

fix b.

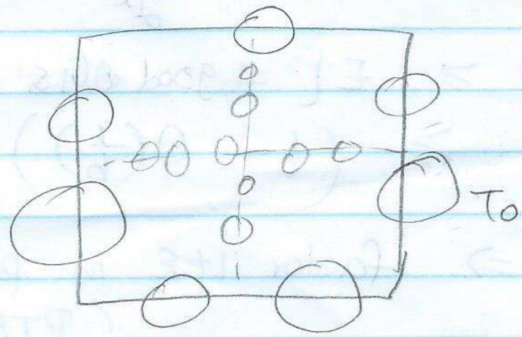
<u>Def</u> An obj of radius r is <u>good</u> if it's inside quadtree cell of side length $\leq br$.

Lemma If all objs are fat & good, problem can be solved in polytime $n^{O(b)}$

Pf: idea – dynamic programming

Subproblem Given a (compressed) quadtree cell B of side length 2^ℓ, & a set T_0 of disjoint objs intersecting ∂B, find size of largest subset T of objs strictly inside B s.t. $T \cup T_0$ is disjoint

"interface"



Note: each obj in T_0 has radius $> \frac{2^\ell}{b}$ (by def'n of good)

$\Rightarrow |T_0| \le O(b)$

$\Rightarrow$ total # subproblems at $B \le n^{O(b)}$

Given sol'n to all subproblems at 4 children of B, can compute sol'n to all subproblems at B by trying all interface combinations

$\Rightarrow$ total time $\left(n^{O(b)}\right)^4 n$

$= n^{O(b)}$ $\square$

Approx Algm:

1. shift all objs by rand vector (in $[0,U]^2$)
2. solve problems on the good objs

Analysis:

Let $t \in T^*$ of radius r, say $2^l \le br < 2^{l+1}$

$Pr[t \text{ not good}] \le Pr[t \text{ crosses grid cell boundary with sidelength } 2^l]$

$$\le 2 \cdot \frac{2r}{2^l} = O\left(\frac{1}{b}\right)$$

$\Rightarrow E[|T|] \ge E[\# \text{ good objs in } T^*]$

$$\ge \left(1 - O\left(\frac{1}{b}\right)\right)|T^*|$$

Set $b = \Theta\left(\frac{1}{\varepsilon}\right) \Rightarrow$ factor $1+\varepsilon$ in poly time $n^{O(1/\varepsilon)}$ (PTAS)

Rmks
- Can be derand.
- works for weighted, arbitrary fat objects
- not work for piercing

Technique 3 : Separators

Planar Graph Separator Thm (Lipton-Tarjan '77)

Given a planar graph $G=(V,E)$, $|V|=n$,

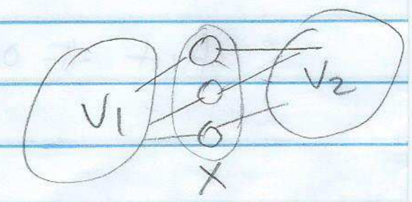
can partition V into V_1, V_2, X s.t.

$\partial V_i \subseteq X$

all neighbors of V_i excluding V_i

$|V_1|, |V_2| \leq \frac{2}{3}n$

$|X| \leq O(\sqrt{n})$

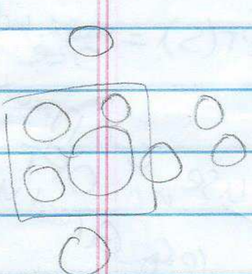


or of const depth

Geometric Separator Thm (Smith-Wormald '98)

Given n disjoint disks / squares / fat objs, in $\mathbb{R}^2$,

$\exists$ square B s.t.



(or disk, or...)

\# objs inside $B \leq \frac{4}{5}n$

\# objs outside $B \leq \frac{4}{5}n$

\# objs intersecting $\partial B \leq O(\sqrt{n})$. $\quad O(n^{1-1/d})$ in $\mathbb{R}^d$

[extends planar graph case]

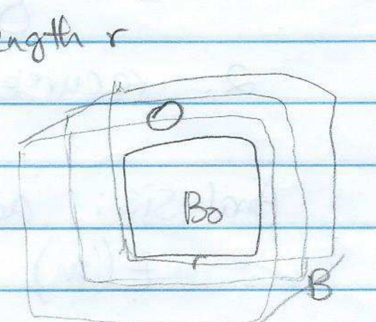
Pf: Let B_0 be smallest square containing $\geq \frac{n}{5}$ center pts

side length r

Let B be a randomly shifted

square of side length $(2-\delta)r$

containing B_0



$\frac{n}{5} \leq$ \# center pts inside $B \leq \frac{4}{5}n$

(since B can be covered by 4 squares of side length $< r$.)

Fix an obj s of radius $\le r/k$.

$\Rightarrow \Pr[\, s \text{ intersects } \partial B \,] \le \frac{r/k}{r} = \frac{1}{k}$.

$\Rightarrow E[\, \#\text{objs of radius} \le r/k \text{ intersecting } \partial B \,] \le \frac{n}{k}$.

But # objs of radius $> r/k$ intersecting $\partial B \le O(k)$
by fatness & disjointness
or const depth

$\Rightarrow E[\, \#\text{objs intersecting } \partial B \,] \le O\left(\frac{n}{k} + k\right)$
$= O(\sqrt{n})$
by setting $k = \sqrt{n}$. □

Fix b.
let OPT(S) = max indep set size

Indep set for disks/squares/fat objs: [C'01]

$O(n^{b})$ time → 0. if $\text{OPT}(S) \le b$ then brute force

1. find square B s.t.

by modifying above proof

$\text{OPT}(\text{objs inside } B) \ge \frac{\text{OPT}(S)}{5c}$ ← use c-approx on OPT (e.g. Greedy).

$\text{OPT}(\text{objs outside } B) \ge \frac{\text{OPT}(S)}{5c}$

$\text{OPT}(\text{objs intersecting } \partial B) = O(\sqrt{\text{OPT}(S)})$

2. recurse on {objs inside B}, {objs outside B}

Analysis: additive error

$$E(m) \le \begin{cases} E(m_1) + E(m_2) + O(\sqrt{m}) & \text{if } m > b \\ \quad \text{for some } m_1, m_2 \ge \frac{m}{5c}, \\ \quad m_1 + m_2 \le m \\ 0 & \text{if } m \le b. \end{cases}$$