Geometric NP-Hard Problems

Independent Set

Given $n$ objects $S$, find max subset $T^* \subseteq S$
$s.t. no 2 objects in $T^*$ intersect

(objects: unit disks/squares/balls, intersection graph
arb. disks/squares/rectangles, line segs,
triangles, polygons, ...)

(Koekoek's Thm: planar graphs are contact graphs of disks)

Hitting Set / "Piercing" (Continuous Version)

Given $n$ objects $S$, find min # pts $X^*$
that stab all objects

(Obs: piercing # $\geq$ independence #)

Hitting Set (Discrete Version)

Given $n$ objects $S$, $m$ pts $P$,
find min subset $X^* \subseteq P$ that stabs all of $S$
(unweighted vs. weighted)

Set Cover (Discrete Version)

Given $n$ pts $P$, $m$ objects $S$
find min subset $T^* \subseteq S$ that covers all of $P$.  
(equiv to hitting set in dual range space
for unit disks, equiv to hitting set)

Set Cover (Cont. Version)

Given $n$ pts $P$,
find min # objects that cover all of $P$.
Dominating Set

Given \( n \) objects \( S \), find \( \min T^* \subseteq S \) s.t.

- every object in \( S - T^* \) intersects some object in \( T^* \)

(reduces to set cover in a certain range space)

(for unit disk/square, equivalent to set cover)

Technique 0: Greedy

indep set

\( T = \emptyset \)

repeat \{ in radius

\[ \text{polytime} \]

pick smallest object \( t \in S \)

insert \( t \) to \( T \), remove \( t \) & all objects intersecting \( t \)

\[ \]

from \( S \).

Analysis:

For squares,

\[ \text{every square intersecting } t \]

\[ \text{bigger than } t \]

is stabbed by a corner of \( t \)

Let \( X^* \) be min piercing set

\[ \Rightarrow |X^*| \leq 4|T^*| \]

Recall \( |T^*| \leq |X^*| \)

\[ \Rightarrow |T^*| \leq 4|T^*| \Rightarrow \text{factor } 4. \]
For disks,

\[ \text{every disk intersecting } t \text{ bigger than } t \text{ can be stabbed by } \leq 16 \text{ pts} \]

\[ |T^*| \geq 16 |T| \implies \text{factor } \leq 16 \]

For "fat objects", const factor

Remarks:
- not work for arbitrary rectangle;
- works for piercing
  (e.g., for squares, \( \ell x \leq 4 |T| \leq 4 \sqrt{\pi} |T| \leq 4 |T| \))
- not work for weighted arb. squares

Technique: Shifted Grid (Hochbaum-Maass '85)

Indep set for unit disks/squares or "fat objects of similar size" (\( \max r/\min r = 0(1) \))

1. Shift all objects by rand. vector (in \([0, 1]^2\))
2. Form grid of side length \( b \)
3. Remove all objects that cross grid cell boundaries
4. Solve problem inside each grid cell by brute force

\[ \text{Solve has } \leq O(b^2) \text{ objects} \]

\[ \text{Runtime: } O(\sum_i^c n_i) = n O(b^2). \]
Analysis:

Fix \( t \in T^* \).

\[ \text{Prob}[t \text{ crosses grid boundary}] \leq \frac{2}{b} \]

\[ \Rightarrow E[1_{T^*}] \geq E[\text{# objs in } T^* \text{ not cross grid boundary}] \geq (1 - \frac{2}{b}) |T^*| \]

Set \( b = \Theta(\frac{1}{\epsilon}) \Rightarrow (1 + \epsilon)\)-approx in time \( O(1/\epsilon^2) \)

Poly-time approx. scheme (PTAS)

Remark: can be derandomized, works for weighted
- works for piercing for fat objects of similar size
- not work for cont. urs. of hitting set

What about arbitrary disks/squares?

Technique 2: Shifted Quadtrees (Erlebach-Jansen-Seidel 01/C. 01)

Indep set for arb. disks/squares, or fat objects

Fix \( b \).

Def: \( A \in \mathcal{L}_b^r \) if radius \( r \) is good
its inside quadtree cell of side length \( \leq br \).
Lemma: If all objects are good, the problem can be solved in polytime $O(b)$.

Proof: Idea - dynamic programming.

Subproblem: Given a quadtree cell $B$ of side length $2^k$, & a set $T_0$ of disjoint objects intersecting $dB$, find size of largest subset $T$ of $T_0$ such that $T \cup T_0$ is disjoint.

Note: Each object in $T_0$ has radius $> \frac{2^k}{b}$ (by definition of good).

$|T_0| \leq O(b)$

$\Rightarrow$ total # subproblems at $B \leq n O(b)$

Given solution to all subproblems at 4 children of $B$, can compute solution to all subproblems at $B$ by trying all interface combinations.

$\Rightarrow$ total time $(nO(b))^4 = n O(b)$ \qed
Approx Algm:
1. Shift all objs by rand vector (in $[0,T]^2$)
2. Solve problems on the good objs

Analysis:
Let $t \in T^*$ of radius $r$, say $2^l \leq br < 2^{l+1}$
$Pr[t \not \text{ not good}] \leq Pr[t \text{ crosses grid cell boundary with sidelength } 2^l]$
\[ \leq 2 \cdot 2r \cdot \frac{2^l}{2^l} = O\left(\frac{1}{b}\right) \]
\[ \Rightarrow E[|t|] \geq E[\# \text{ good objs in } T^*] \geq \left(1 - O\left(\frac{1}{b}\right)\right) |T^*| \]

Set $b = \Theta(\frac{1}{\varepsilon}) \Rightarrow$ factor $1 + \varepsilon$ in poly time $n^{O(1/\varepsilon)}$ (PTAS)

- can be derand.

Runs - works for weighted, arbitrary fat objects
- not work for piercing
Technique 3: Separators

Planar Graph Separator Thm (Lipton-Tarjan '77)
Given a planar graph $G = (V, E)$, $|V| = n$, can partition $V$ into $V_1, V_2, X$ st.
\[ |V_i| \leq X \]
\[ \forall \text{ all neighbors of } V_i, \text{ excluding } V_i. \]
\[ |V_1|, |V_2| \leq \frac{2}{3} n \]
\[ |X| \leq O(\sqrt{n}) \]
or of const depth

Geometric Separator Thm (Smith-Wormald '98)
Given $n$ disjoint disks/squares/fat obs, in $\mathbb{R}^2$,
\[ \exists \text{ square } B \text{ st.} \]
\[ \# \text{ objs inside } B \leq \frac{4}{5} n \]
\[ \# \text{ objs outside } B \leq \frac{4}{5} n \]
\[ \# \text{ objs intersecting } \partial B \leq O(\sqrt{n}) \]
[extends planar graph case]

Pf: Let $B_0$ be smallest square containing $\geq \frac{n}{5}$ center pts
Let $B$ be a randomly shifted square of side length $(2-\delta) r$
containing $B_0$
\[ \frac{4}{5} \leq \# \text{ center pts inside } B \leq \frac{4}{5} n \]
(since $B$ can be covered by 4 squares of side length $r$).
Fix an obj s of radius $\leq r/k$.

$\Rightarrow \ Pr [ \text{s intersects } dB] \leq \frac{r/k}{r} = \frac{1}{k}$.

$\Rightarrow E[\# \text{objs of radius } r/k \text{ intersecting } dB] \leq \frac{n}{k}$.

But $\# \text{objs of radius } > r/k \text{ intersecting } dB \leq O(k)$

by fatness & disjointness
or const depth.

$\Rightarrow E[\# \text{objs intersecting } dB] \leq O(\frac{k}{k} + k)$

$= O(\sqrt{n})$

by setting $k = \sqrt{n}$.

Indep set for disks/squares/fat obj s: $[C_{01}]$

$O(b)$ time $\Rightarrow O(.)$ if $OPT(S) \leq b$ then brute force

1. find square $B$ s.t.

by modifying above proof

$OPT(\text{objs inside } B) \geq \frac{OPT(S)}{5c}$

$OPT(\text{objs outside } B) \geq \frac{OPT(S)}{5c}$

$OPT(\text{objs intersecting } dB) = O(\sqrt{OPT(S)})$

2. recurse on $\{\text{objs inside } B\} , \{\text{objs outside } B\}$

Analysis: additive error

$E(m) \leq \left\{ \begin{array}{ll}
E(m_1) + E(m_2) + O(\sqrt{m}) & \text{if } m > b \\
n & \text{if } m \leq b
\end{array} \right.$

from $m_1, m_2 \Rightarrow \frac{m}{5c}$.

$m_1 + m_2 \leq m$.

$O(.)$ if $m \leq b$. 