Homework 2 (due Oct 17 Wednesday (11am in class))

Instructions: You may work in groups of at most 2. Hand in one set of solutions per group. Acknowledge any discussions you have with other students and other sources you have consulted. Solutions must be written *in your own words*.

1. [*20 pts*] Smith and Wormald proved the following geometric separator theorem (which we will cover in a future lecture): given a set S of n squares in two dimensions, there exists a new square t such that the number of squares strictly inside t is at most $0.8n$, the number of squares strictly outside t is at most $0.8n$, and the numer of squares intersecting the boundary of t is at most $O(\sqrt{n})$.

   Assuming that the theorem is true (you don't need to know the proof), give a randomized algorithm to compute a square t' such that the number of squares strictly inside t' is at most $0.9n$, the number of squares strictly outside t' is at most $0.9n$, and the number of squares intersecting the boundary of t' is at most $O(n^{0.99})$. Your algorithm should run in *sublinear* time and has high probability of correctness.

   Hint: use ε-approximations for an appropriate range space (how should we choose ε?).

2. [*30 pts*] We want to devise an efficient algorithm for computing a $(1+\varepsilon)$-factor approximation for the following Best-3-Sided-Rectangle problem:

   > Given a set P of n points in two dimensions, find a 3-sided rectangle, with area equal to 1, that maximizes the number of points of P inside the rectangle.

   Here, a *3-sided rectangle* refers to a rectangle whose bottom side lies on the x-axis.

   (a) [*10 pts*] First prove the following lemma: given any $1 \leq k \leq n$, we can partition the plane into a collection Γ of $O(n/k)$ disjoint rectangles, such that each rectangle of Γ contains at most k points and has exactly k points in its shadow (except for the bottommost rectangle). Here, the *shadow* of a rectangle $[a,b] \times [u,v]$ refers to the 3-sided rectangle $[a,b] \times [0,u]$. The partition should be constructible in $O(n \log n)$ time.
   Hint: start at the bottom, with a rectangle covering the lowest $2k$ points, then move up recursively...
   (For the running time, you may use the following known fact: any array of n numbers can be preprocessed in a data structure in $O(n)$ time so that the k smallest elements in any subarray can be found in $O(k)$ time.)

   (b) [*8 pts*] As a corollary of (a), prove that there exists a collection Γ' of $O(n/k)$ 3-sided (but not disjoint) rectangles, each containing at most $O(k)$ points of P, such that any arbitrary 3-sided rectangle containing at most k points of P is contained inside some rectangle of Γ'.
   Hint: for every pair of adjacent rectangles in Γ, create a new 3-sided rectangle in Γ'...

(c) [*12 pts*] Using (b), describe a $(1+\varepsilon)$-factor approximation algorithm for the Best-3-Sided-Rectangle problem that runs in $O(n \log^{O(1)} n)$ time (with high probability of correctness, if randomization is used).