Spectral Graph Drawing

based on papers by Yehuda Koren

Presenter: Tana Wattanawaroon

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The Graph Drawing Problem

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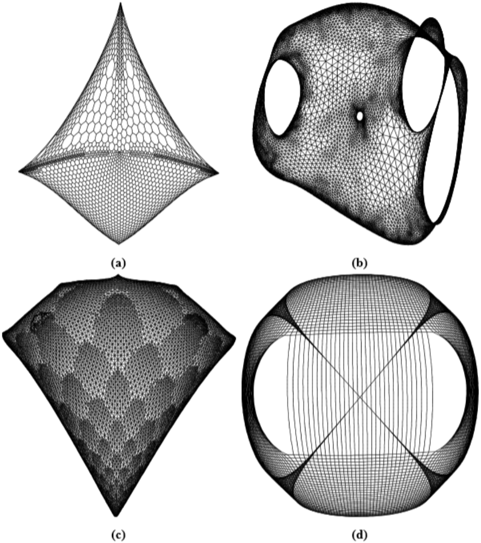
- Draw edges using only straight segments.
- Goal: place the vertices in p-dimensional space to get a "beautiful" layout.

The Graph Drawing Problem



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- vertices connected by an edge to be close to each other
- higher weighted edge: even closer
- but not too close in general

Formulation

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- A p-dimensional layout defined by $x^1, \dots, x^p \in \mathbb{R}^n$.
- Vertex i is at coordinates $(x^1(i), \dots, x^p(i))$.
- Euclidean distance between vertices i and j

$$d_{ij} = \sqrt{\sum_{k=1}^{p} (x^k(i) - x^k(j))^2}$$

Spectral answers to optimization problems

Theorem (1)

Given a symmetric matrix $A_{n\times n}$, denote by $v^1, \ldots, v^n$ its eigenvectors, with corresponding eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. Then, $v^1, \ldots, v^p$ are an optimal solution of the constrained minimization problem:

$$\min_{x_1,\ldots,x_p} \sum_{k=1}^{p} (x^k)^T A x^k$$

subject to

$$(x^k)^T x^l = \delta_{kl}, \quad k, l = 1, \ldots, p$$

where δ_{kl} is 1 if $k = l$, 0 otherwise.

Spectral answers to optimization problems

Generalized eigenvector of (L, D) is u s.t.

$$Lu = \mu Du$$

Spectral answers to optimization problems

Theorem (2)

Given a symmetric matrix $A_{n\times n}$ and a positive definite matrix $B_{n\times n}$, denote by $v^1,\ldots,v^n$ the generalized eigenvectors of (A,B), with corresponding eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. Then, $v^1,\ldots,v^p$ are an optimal solution of the constrained minimization problem:

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$$(x^1)^T B x^1 = (x^2)^T B x^2 = \cdots = (x^p)^T B x^p$$
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Spectral answers to optimization problems

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$$(x^i)^T B v^j = 0$$

"Force-Directed" Method

- Minimization problem:

$$\min_{x_1,\ldots,x_p} \frac{\sum_{(i,j)\in E} w_{ij} d_{ij}^2}{\sum_{i<j} d_{ij}^2}$$

$$\text{subject to } \operatorname{Var}(x^1) = \ldots = \operatorname{Var}(x^p)$$
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- Thus we can reformulate the problem as

$$\min_{x_1,\ldots,x_p} \frac{\sum_{k=1}^{p}(x^k)^T L x^k}{\sum_{k=1}^{p}(x^k)^T x^k}$$

$$\begin{aligned} \text{subject to } \quad & (x^k)^T(x^l) = \delta_{kl}, \qquad && k, l = 1, \ldots, p \\ \text{and } \quad & (x^k)^T \cdot 1_n = 0, && k = 1, \ldots, p \end{aligned}$$

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- The answer is $v^2, \ldots, v^{p+1}$.

Degree-Normalized "Force-Directed" Method

- Minimization problem:

$$\min_{x_1,\ldots,x_p} \frac{\sum_{k=1}^{p}(x^k)^T L x^k}{\sum_{k=1}^{p}(x^k)^T D x^k}$$

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- The answer is $u^2,\dots,u^{p+1}$, the generalized eigenvectors of (L,D).

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- Differentiating $x^T Lx$ with respect to $x(i)$ gives

$$\frac{\partial x^T Lx}{\partial x(i)} = 2 \sum_{j \in N(i)} w_{ij}(x(i) - x(j))$$

which is zero when

$$x(i) = \frac{\sum_{j \in N(i)} w_{ij} x(j)}{\deg(i)}$$

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- This can be written, for all i together, as

$$D^{-1} L x = \mu x$$
$$L x = \mu D x$$

which endorses the use of generalized eigenvectors of (L, D)

Weighted Centroid Characterization

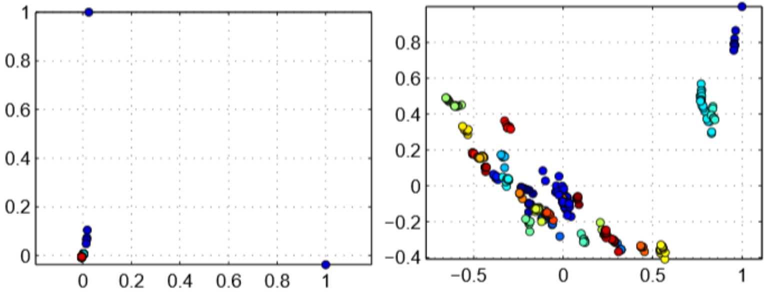
- Using generalized eigenvectors of (L, D) gives

$$x(i) - \frac{\sum_{j \in N(i)} w_{ij} x(j)}{\deg(i)} = \mu \cdot x(i)$$

- Using Laplacian eigenvectors gives

$$x(i) - \frac{\sum_{j \in N(i)} w_{ij} x(j)}{\deg(i)} = \lambda \cdot \deg(i)^{-1} \cdot x(i)$$

Weighted Centroid Characterization



(a) (b)

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- Random walk!

References

- Koren, Yehuda. "Drawing graphs by eigenvectors: theory and practice." *Computers & Mathematics with Applications* 49.11 (2005): 1867-1888.
- Koren, Yehuda. "On spectral graph drawing." *Computing and Combinatorics*. Springer Berlin Heidelberg, 2003. 496-508.