CS 598: Spectral Graph Theory. Lecture 17

Effective Resistance

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Today

- Resistance in networks and motivation form Physics.
- Ohm, Kirchoff and Laplacian.
- Effective Resistance.
- Linear Equations in Laplacians.

Resistor Networks

- One of our main motivations for studying the Laplacian matrix is its role in the analysis of physical systems.
- Given graph, we treat each edge as resistor.
- If graph unweighted, resistance is 1.
- Generally, resistance is $\frac{1}{w(e)}$.
- The smaller the weight, the larger resistance (if no edge, the infinite resistance).

Resistor Networks

- **Ohm's Law**: Potential drop across a resistor is equal to the current flowing over the resistor times the resistance

  V=IR

- For graph, define for each edge (a,b) the current flowing from a to b as

  $i(a, b) = -i(b, a)$

- If $v \in R^n$ is the vector of potentials, then $i(a, b) = w(a, b) \cdot (v(a) - v(b))$ (flowing from high to low potential)

Resistor Networks

- We want to write the equation in matrix form.
- Treat i as a vector with entries for every edge (a,b) where a<b.
- Recall the edge-vertex adjacency matrix

$$U((a,b),c)\begin{cases} 1\ if\ a=c \\ -1\ if\ b=c \\ 0\ o.w \end{cases}$$

- If W is the matrix with weights of edges in diagonal, then $i = WUv$.

Resistor Networks

- **Kirchoff's Law:** Resistor networks cannot hold current. All flow entering from vertex a from edges in the graph must exit a to an external source.
- Let $i_{ext} \in R^n$ denote the external currents, where $i_{ext}(a)$ is the amount of current entering the graph through node a.
- $i_{ext}(a) = \sum_{b:(a,b)\in E} i(a,b)$

Resistor Networks

- In matrix form

$$i_{ext} = U^T i = U^T W U v = L v$$

- External nodes when $i_{ext} \neq 0$, internal nodes when its zero.
- For internal nodes and unweighted graph, this equation implies:

$$0 = L(a,\cdot)v = \sum_{(a,b)\in E}(v(a) - v(b)) = d(a)v(a) - \sum v(b)$$

- So voltages are weighted average of neighbors.

Resistor Networks

- We would like to apply the equation in reverse, and find the potentials from the currents.

$$i_{ext} = Lv = L^{-1} i_{ext} \Rightarrow v$$

- Laplacian has no inverse!!
- Define Pseudoinverse, since we are only interested in currents that sum to zero.

Resistor Networks

- **Definition** (Pseudoinverse).

$$LL^+ = \Pi$$

Where Π is symmetric projection onto span of L.

- **Claim.** $L^+ = \sum_{i>1} \frac{1}{\lambda_i} v_i v_i^T$
- In general, for every rational function of the laplacian…

The Path Graph

- Example: the path with unit flow.

Effective Resistance

- Effective resistance between vertices a and b is the resistance between a and b given by the whole network, if we treat it as a resistor.
- Recall serial and parallel composition.
- $i(a,b) = \frac{v(a)-v(b)}{r_{a,b}}$. Define $R_{eff}(a,b)$ to be the potential different between a and b if we send one unit of current through a and remove it from b.

Effective Resistance

- Define

$$i_{ext}(c) = \begin{cases} 1 \; if \; c = a \\ -1 \; if \; c = b \\ 0 \; o.w \end{cases}$$

- This corresponds to a flow of 1 from a to b.
- Solve for voltage as before: $Lv = i_{ext} \Rightarrow v = L^{+} i_{ext} \Rightarrow v(a) - v(b) = i_{ext}^{T} v = i_{ext}^{T} L^{+} i_{ext}$, the effective resistance.
- We can shift by multiple of all-1's vector.

Effective Resistance

- **Claim**. $\mathrm{R_{eff}}(\mathrm{a}, \mathrm{b}) = \mathrm{i_{ext}^T} L^+ i_{ext} = v^T L v$
- **Claim**. If a,b the external nodes and every other node c is internal, then $v(a) \geq v(c) \geq v(b)$.

- Using this claim, show triangle inequalities. Effective Resistance is a distance.

Fixing Potentials

- We can see effective resistance by fixing potentials.
- Solve linear equations in Laplacians to find the potentials.

- Rest of class: properties of submatrices of Laplacian (see board).

Positive Inverse

- We will show the following claims:

- **Claim.** Let A be a symmetric matrix with evals $-1 < \mu_1 \leq \cdots \leq \mu_n < 1$. Then

$$(I - A)^{-1} = \sum_j A^j$$

Positive Inverse

- We will show the following claims:

- **Claim**. Let L=D-A the Laplacian of a connected graph. Let X be a diagonal, non-negative, non-zero matrix. Then
  - $D - A + X$ is positive definite
  - $D + A + X$ is positive definite
  - $(L + X)^{-1}_{ij} > 0$ for all i,j.