



CS 598: Spectral Graph Theory. Lecture 14

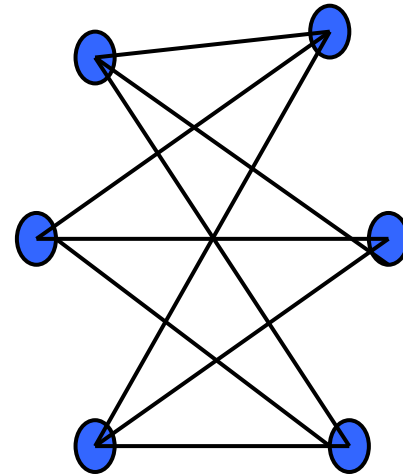
Spectral Algorithms for
Unique Games

Alexandra Kolla

The MAX CUT Problem

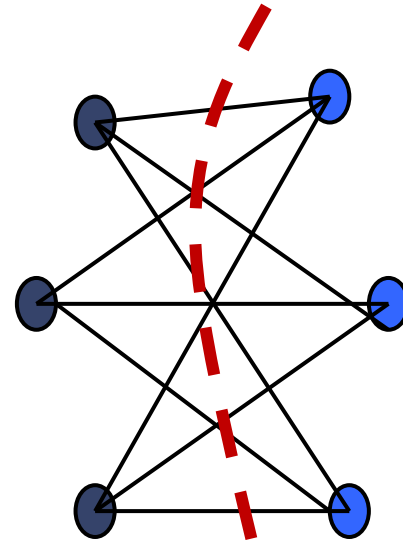
- **Input:** $G = (V, E)$

G



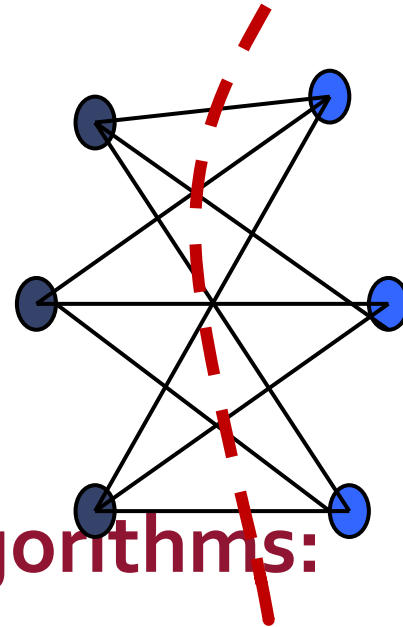
The MAX CUT Problem

- **Input:** $G = (V, E)$
- **Objective :** Partition G in (S, S') as to **MAXIMIZE** number of edges cut
- **[Karp '72]:** MAX CUT is NP-complete
- What about approximating MAX CUT?



The MAX CUT Problem

- **Input:** $G = (V, E)$ **G**
- **Objective :** Partition G in (S, S') as to MAXIMIZE number of edges cut



Approximation algorithms:

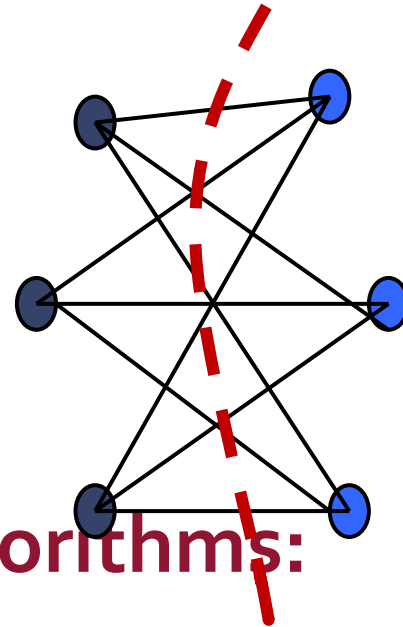
- **Random cut (trivial):** half of optimal
- **[GW'94]:** $\alpha_{GW}=0.878$ approximation algorithm of MAX CUT

How many of you bet this is best we can do?

The MAX CUT Problem

- **Input:** $G = (V, E)$
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Approximation algorithms:

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If UGC True, then it is the best!

Can We Hope for Better Approximation Algorithms in P?

Previous inapproximability not a coincidence!
Unique Games Conjecture (UGC) captures **exact** inapproximability of many more problems

Problem	Best Approximation Algorithm Known	UGC-Hardness
MaxCut	0.878[GW94]	0.878 [KKMO07]
Vertex Cover	2	$2-\epsilon$ [KR06]
Max k-CSP	$\Omega(k/2^k)$ [CMM07]	$O(k/2^k)$ [ST,AM,GR]

What are Unique Games?

I. Unique Games are popular not only among computer scientist!

The screenshot shows a Bing search results page for the query "Unique Games". The search bar at the top contains the text "Unique Games" and a red square icon. Below the search bar, the word "Web" is displayed. On the left side, there is a list of "RELATED SEARCHES" including "Unique Free Online Games", "Unique Puzzle Games", "Unique Family Games", "Unique Party Games", "Unique Board Games", "Unique Golf Games", "Unusual Games", and "Fun Games". The main search results are displayed in the center, showing "ALL RESULTS" for "1-10 of 69,400,000 results". The first result is a sponsored site for "Crate & Barrel" with the URL "www.crateandbarrel.com" and the text "Today Only! Save 15% and Get Free Shipping On Select Orders." The second result is for "Unique Games" from "SpencersOnline.com" with the text "Buy Novelty, Raunchy & Fun Games \$4.99 Shipping on Orders Over \$39!". A red box highlights this second result. To the right of the search results, there is a button that says "Make Bing your homepage" and a section for "Sponsored sites" with two entries: "Uncommon Games" and "unique games".

bing 70 million pages

What are Unique Games?

1. Unique Games are popular not only among computer scientist!

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bing 70 million pages

A screenshot of a Yahoo! search results page for the query "Unique Games". The search bar at the top shows "Unique Games" with a magnifying glass icon. Below the search bar, the text "68,100,000 results for Unique Games" is visible. The results are organized into sections: "Also try:" with links like "baby shower unique games", "unique games online", and "more..."; "Sponsored Results" with "Crate & Barrel" and "SpencersOnline.com"; and "Sponsored Results" with "Uncommon Games" and "unique games". A red box highlights the "Uncommon Games" result, which includes the text: "Find unique, creatively designed board games for adults & teens." and the URL "www.uncommongoods.com".

Yahoo!: 69 million pages

2. We can purchase Unique Games on-line!

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A screenshot of a Bing search results page for the query "Unique Games". The search bar at the top shows "Unique Games" and the Bing logo. Below the search bar, there are several search results. One result, "Unique Games" by agcrump.com, is highlighted with a red box. The text of this result reads: "Card, Arcade, and Board game shareware site. Download a free game title, or link to other game sites." Other results include "Crate & Barrel" and "Uncommon Games".

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bing 70 million pages

3. Unique Games are related to the Unique Games Conjecture...

A screenshot of a Google search results page for the query "Unique Games". The search bar at the top shows "Unique Games" and the Google logo. Below the search bar, there are several search results. One result, "Unique games conjecture - Wikipedia, the free encyclopedia", is highlighted with a red box. The text of this result reads: "In computational complexity theory, the Unique Games Conjecture is a conjecture made by Subhash Khot in 2002. The conjecture postulates the NP-hardness of ...". Other results include "Uncommon Games" and "Unique Games".

Google: 178 million pages

Unique Games = Unique Label Cover Problem

Given: set of constraints

Linear Equations mod k :

$$x_i - x_j = c_{ij} \pmod k$$

GOAL

k = "alphabet" size

Find labeling that satisfies **maximum number of constraints.**

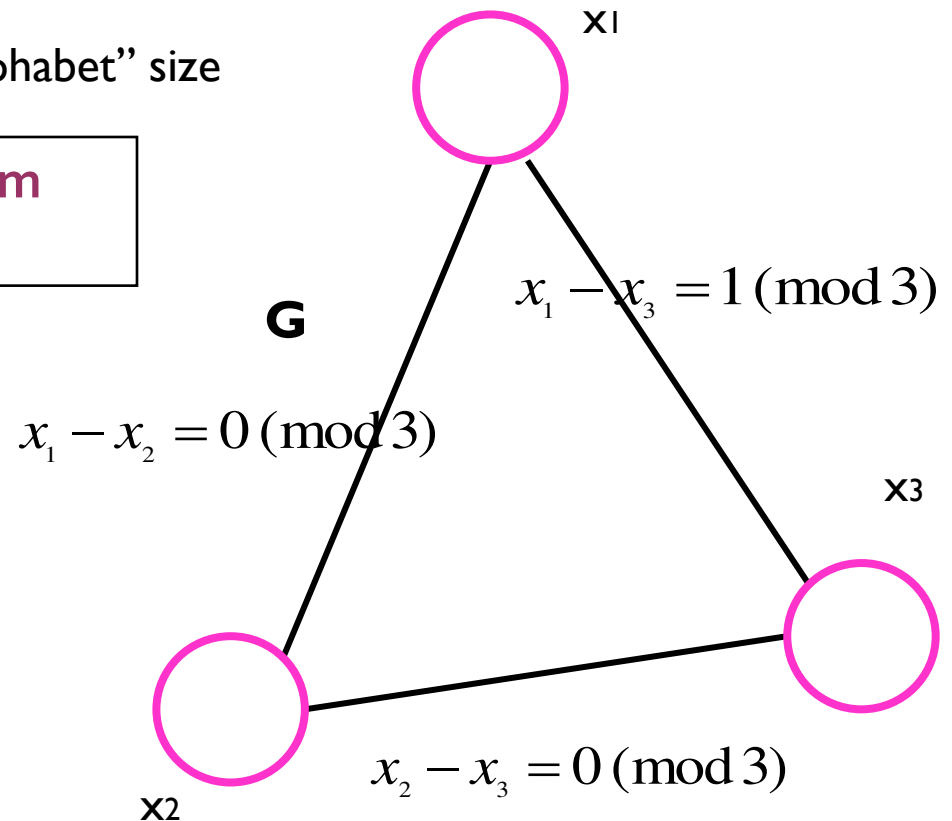
EXAMPLE

$$x_1 - x_2 = 0 \pmod 3$$

$$x_2 - x_3 = 0 \pmod 3$$

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The constraint graph



Unique Games , an Example

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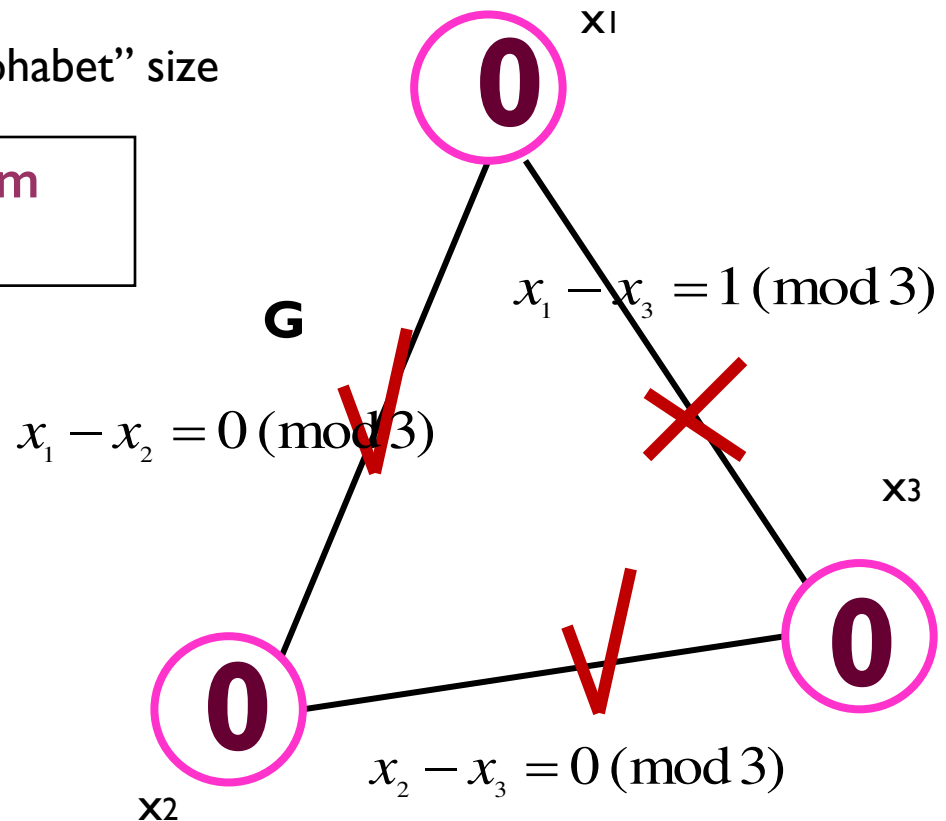
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The constraint graph



Satisfy 2/3 constraints

Unique Games , an Example

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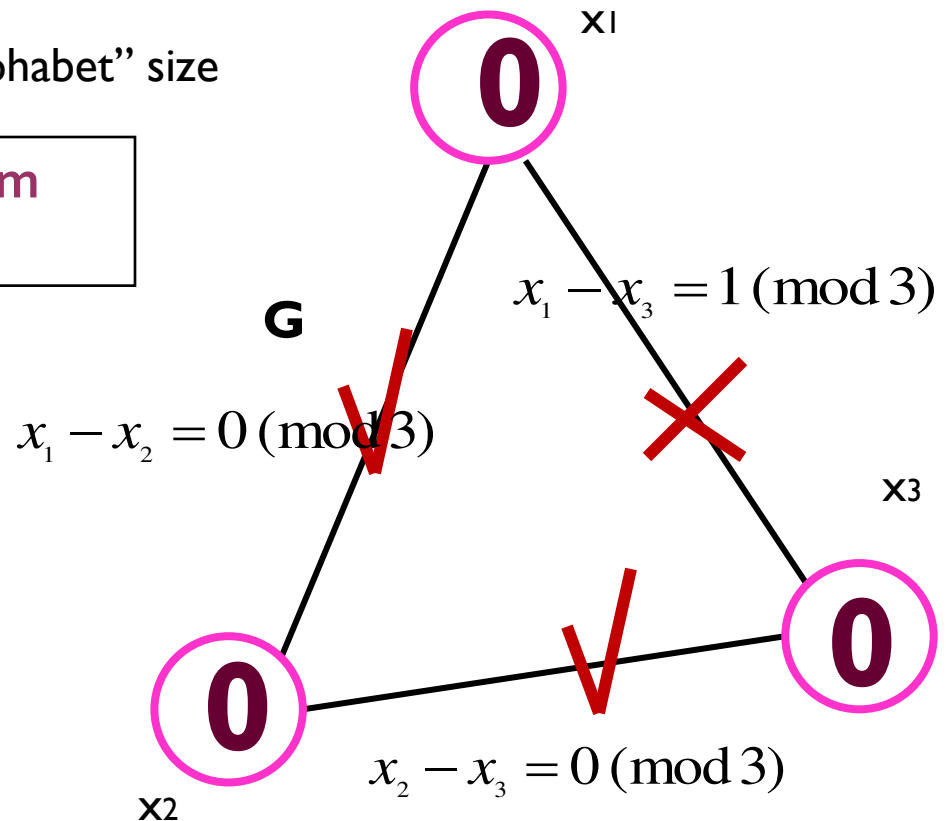
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The constraint graph



Rest of the talk: d -regular graphs

Unique Games Conjecture

- **[Khot'02]** For every positive ε and δ there is a large enough k s.t. for some instance of Unique Games with alphabet size k and $\text{OPT} > 1 - \varepsilon$, it is NP hard to satisfy a δ fraction of all constraints.
- Given UG instance where 99% of constraints are satisfiable, it is NP-hard to even satisfy 0.1%

Unique Games Conjecture

- Embarrassing not to know, since solving systems of linear equations is easy.
- How? (Gaussian elimination, propagation...)

Where to begin if we want to refute UGC?

- Several attempts in recent years to refute or prove UGC.
- Lot of progress but still no consensus.

Plan of attack: start ruling out cases.

- Classify graphs according to their “spectral profile” (eigenvalues)
- Expanders [AKKTSV’08,KT’08],
- Local expanders, graphs with relatively few large eigenvalues [AIMS’09,SR’09,K’10]

- Find distributions that are hard?
 - Random Instances : NO! Follows from expander result.
 - Quasi-Random Instances? [KMM’10] NO!

Summary: Algorithmic Results for UG

General Graphs

Algorithm	On $1-\epsilon$ instances
Khot	$1-O(k^2 \epsilon^{1/5} \sqrt{\log(1/\epsilon)})$
Trevisan	$1-O(\sqrt[3]{\epsilon \log n})$
Gupta-Talwar	$1-O(\epsilon \log n)$
CMM1	$k^{-\epsilon/2-\epsilon}$
CMM2	$1-O(\epsilon \sqrt{\log n} \sqrt{\log k})$

SDP/LP based

Special Graphs

Expander

AKKTSV'08 KT'08, MM'10	Constant, depends on conductance
---------------------------	----------------------------------

Tight for SDP, there is counterexample

Local expander

AIMS'09, SR'09	Constant, depends on local expansion
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Almost all above approaches were LP or SDP based

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Few large eigenvalues

K'10	Quality and running time depends on eigenspace
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Purely SPECTRAL Approach "beats" SDP

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ABS'10: Subexponential time algorithm for ANY instance

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KMM'10: Semi-Random instances are easy

instance

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GOAL

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Find labeling that satisfies **maximum number of constraints.**

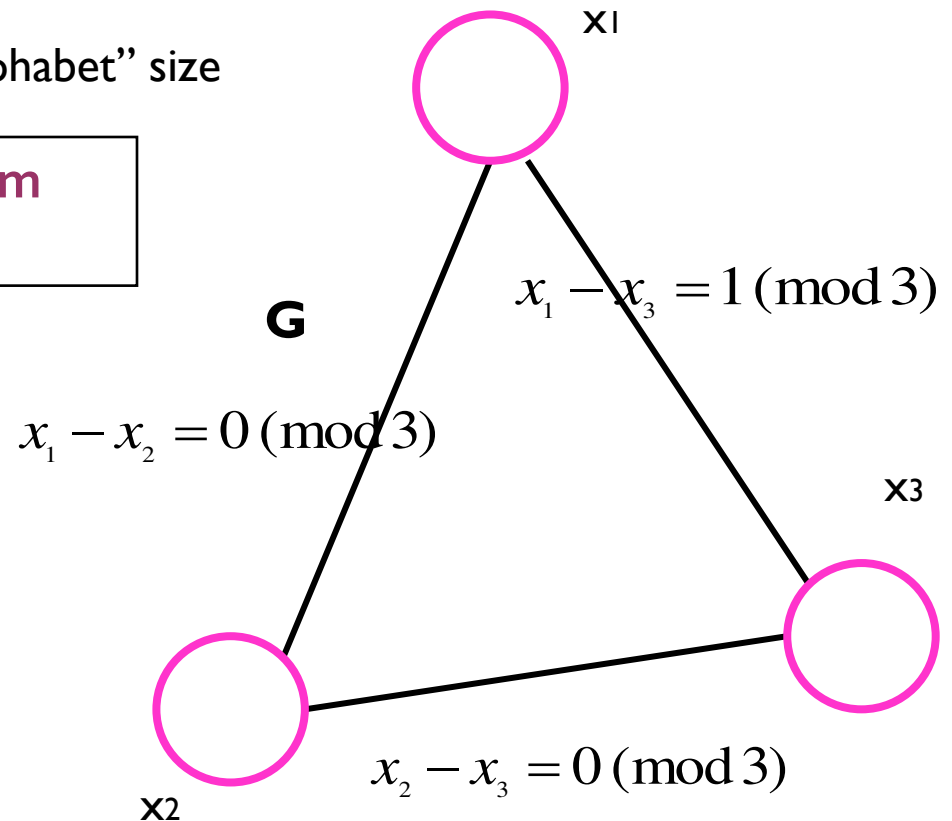
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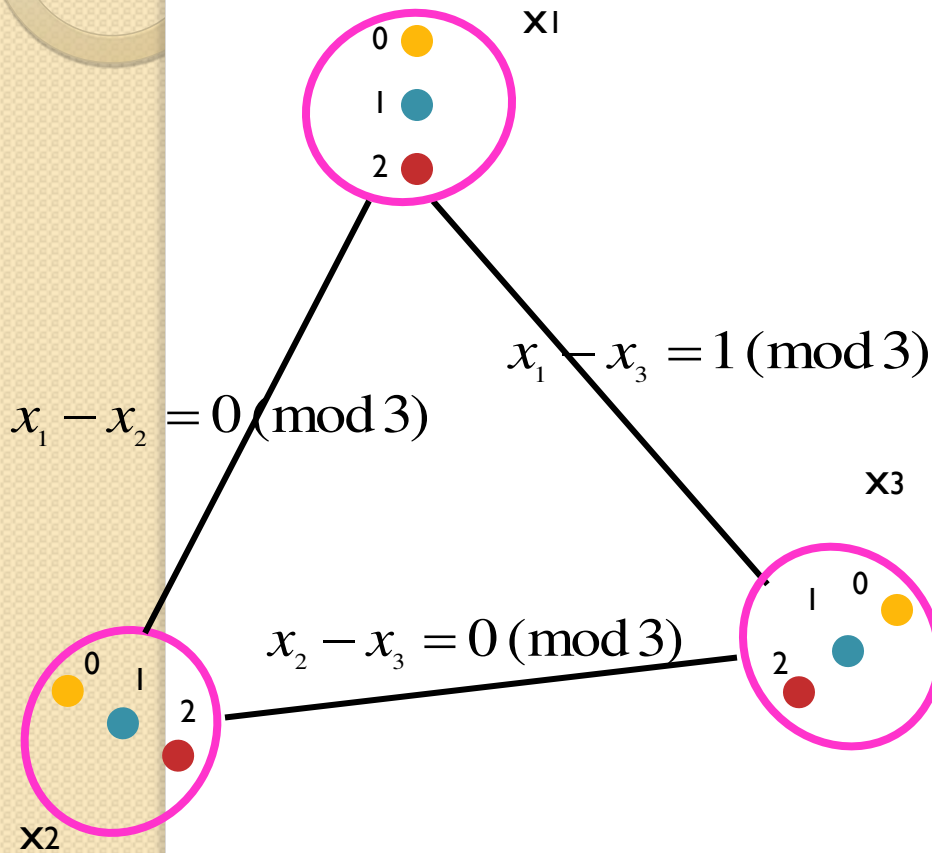
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The constraint graph

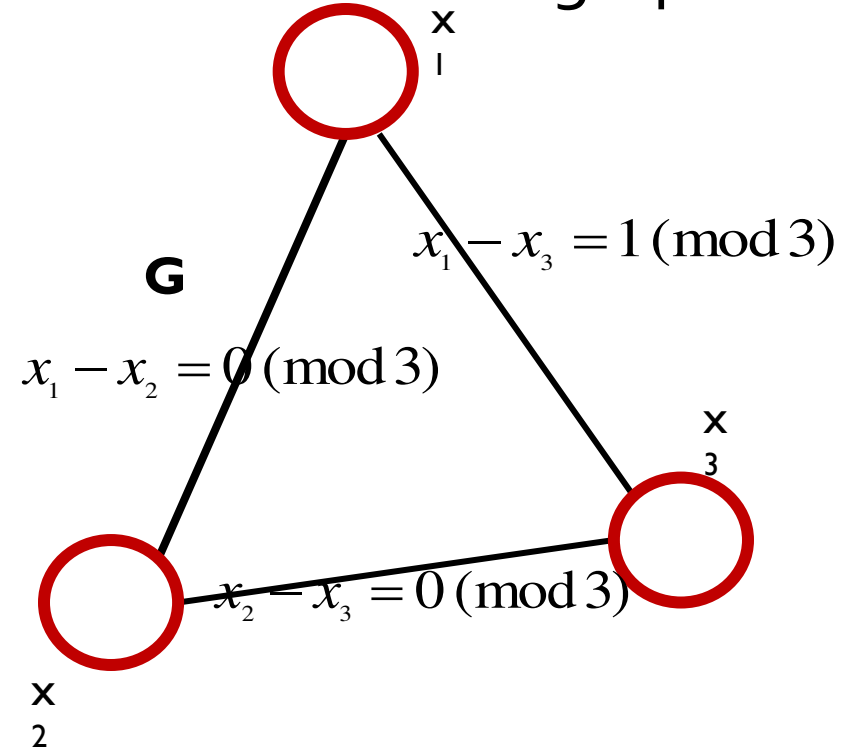


Unique Games and Graphs

2. The "label-extended" graph



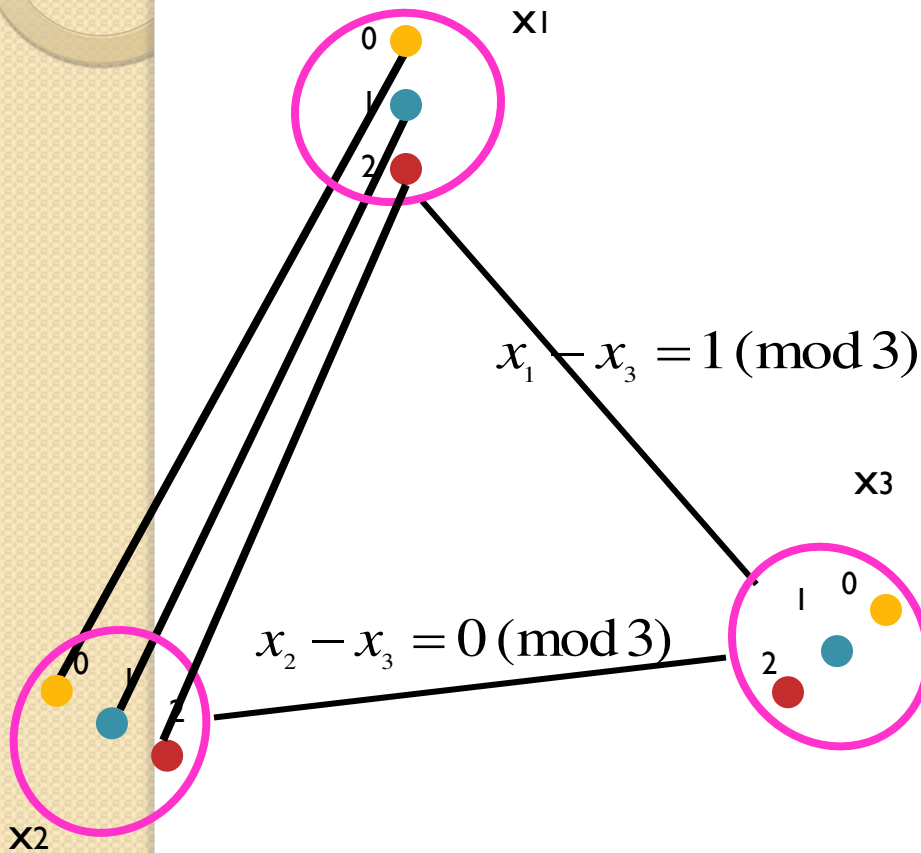
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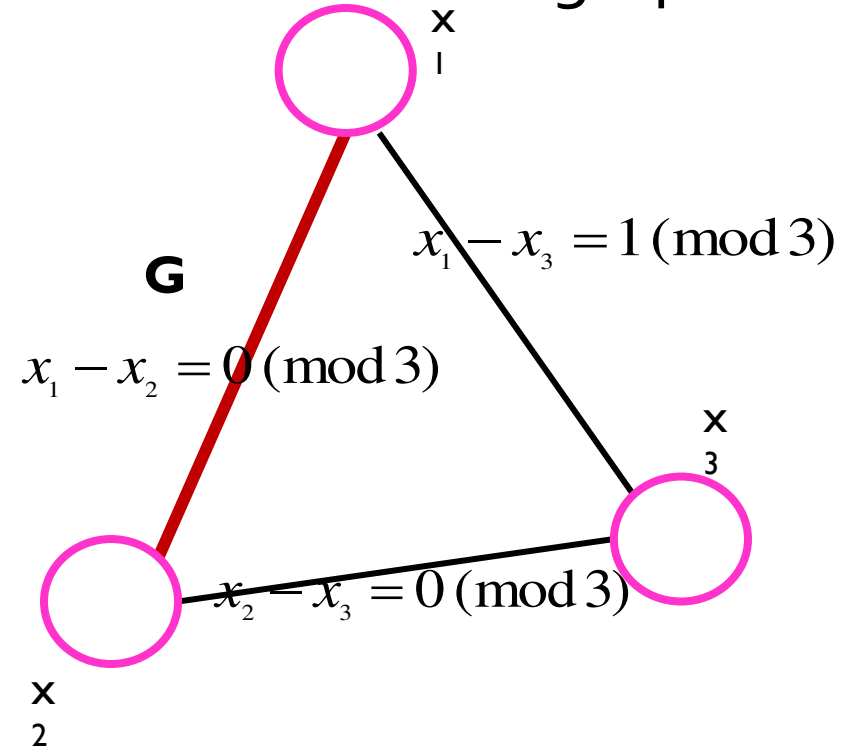
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Unique Games and Graphs

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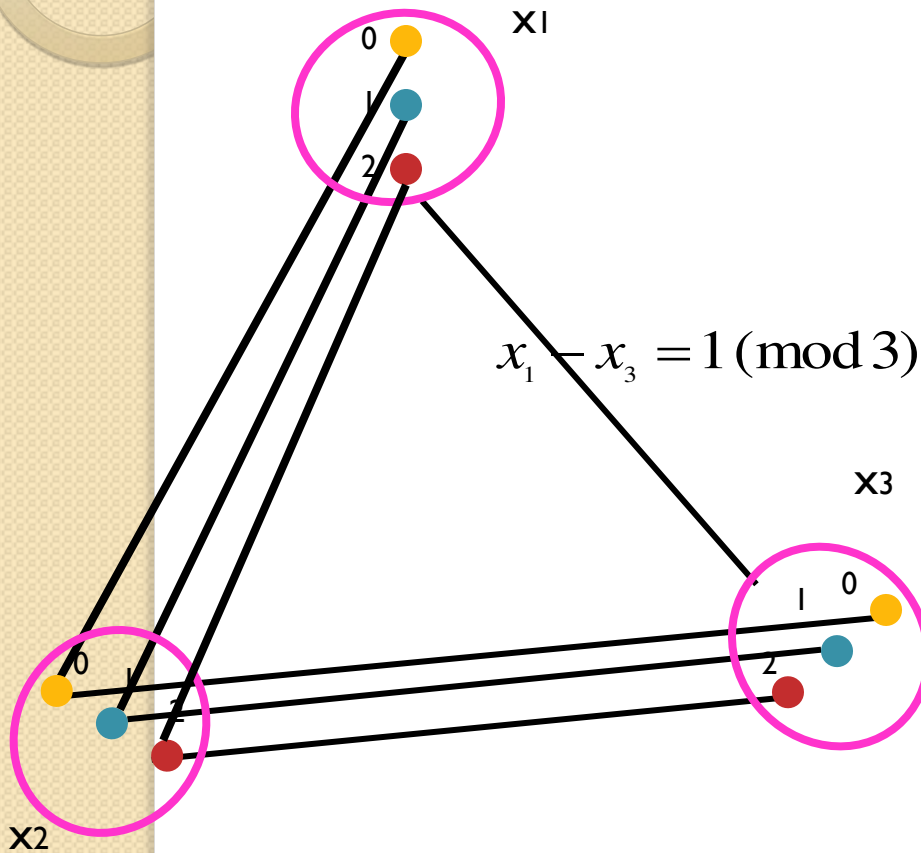


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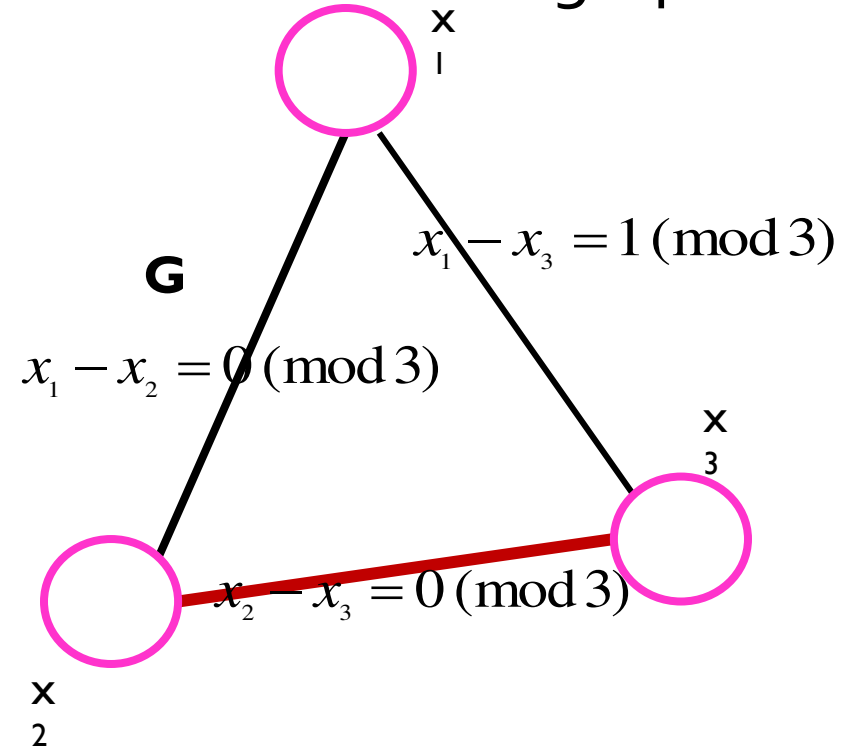
• Replace each edge with the "permutation matching"

Unique Games and Graphs

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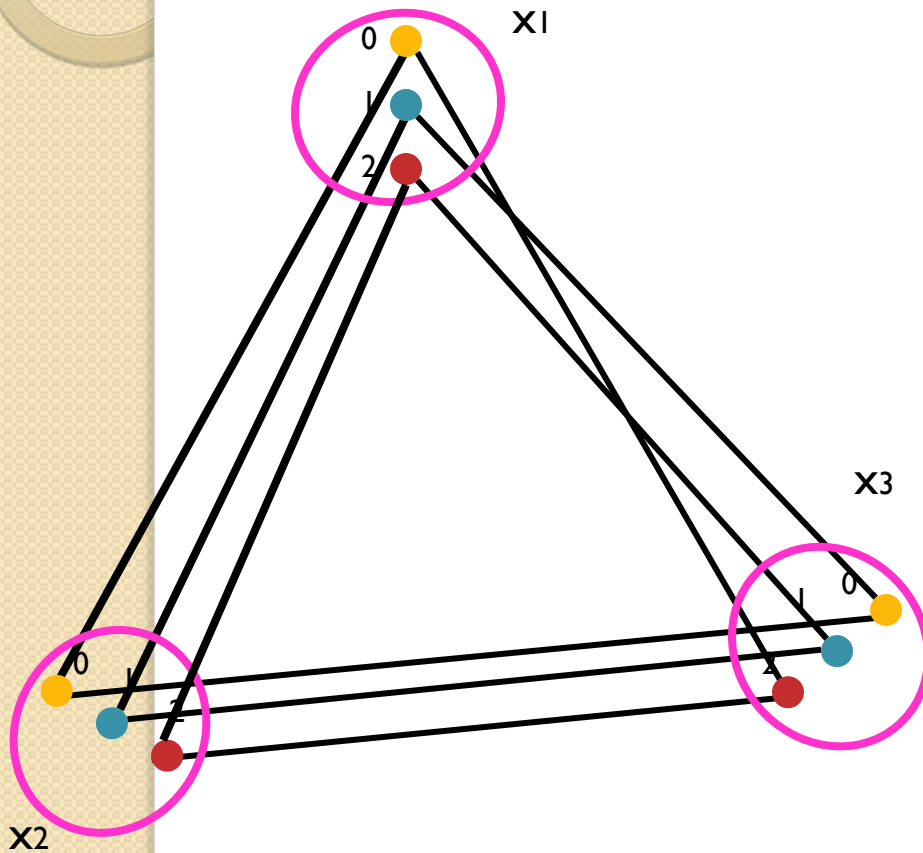


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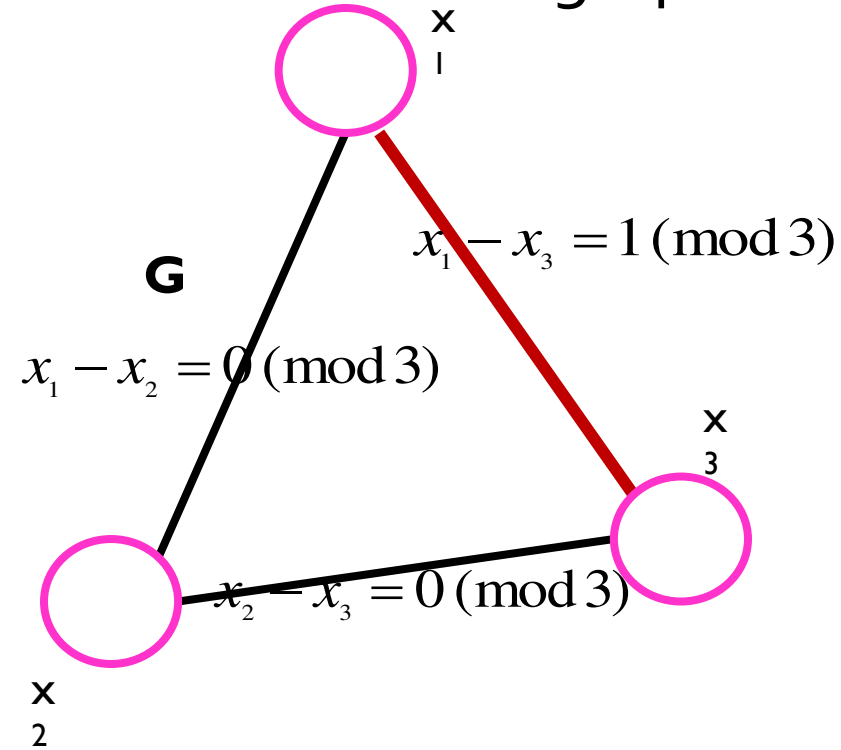
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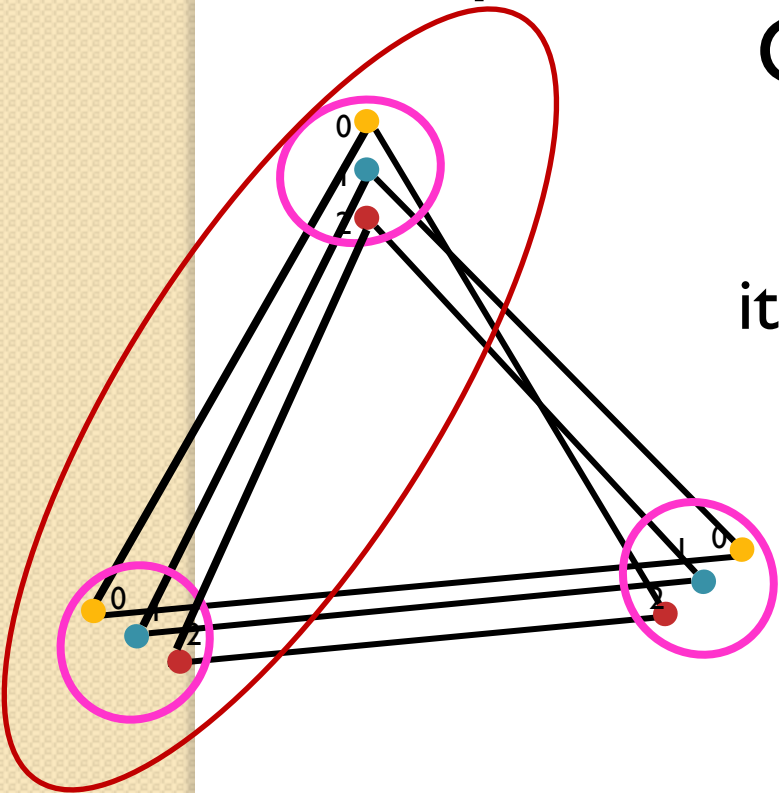
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- Replace each edge with the "permutation matching"

More Graph Theory: The Label-Extended Graph

GRAPH THEORY?

it's a graph, it has adjacency matrix!



0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	0	0	1

M has each non – zero entry (u,w) replaced by a block corresponding to the permutation on edge

Sketch UGC False on Expanders

UGC FALSE on expanders [AKKTSV'08, KT'08 MM'10]:
When UG instance highly satisfiable and **graph is expander**, ptime algorithm finds labeling that satisfies 99% of the constraints

Why Expanders? Expansion of Unique Games and Sparsest Cut

Problem	Best Approximation Algorithm Known	UGC-Hardness
MaxCut	0.878 [GW94]	0.878 [KKMO07]
Vertex Cover	2	$2-\epsilon$ [KR06]
Max k-CSP	$\Omega(k/2^k)$ [CMM07]	$O(k/2^k)$ [ST,AM,GR]

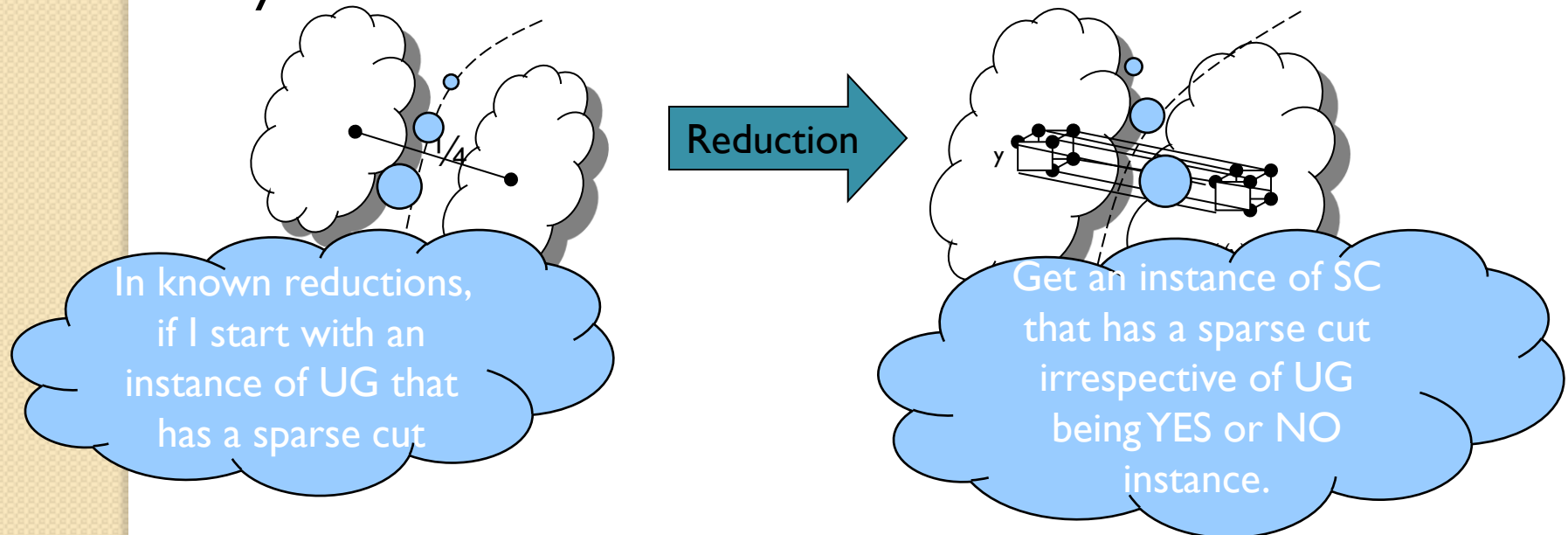
Uniform
Sparsest

No hardness even assuming
UGC unless expansion

Why Expanders? Expansion of Unique Games and Sparsest Cut

No hardness for Sparsest Cut even assuming UGC!

Unlikely that there is reduction from UG to SPARSEST CUT

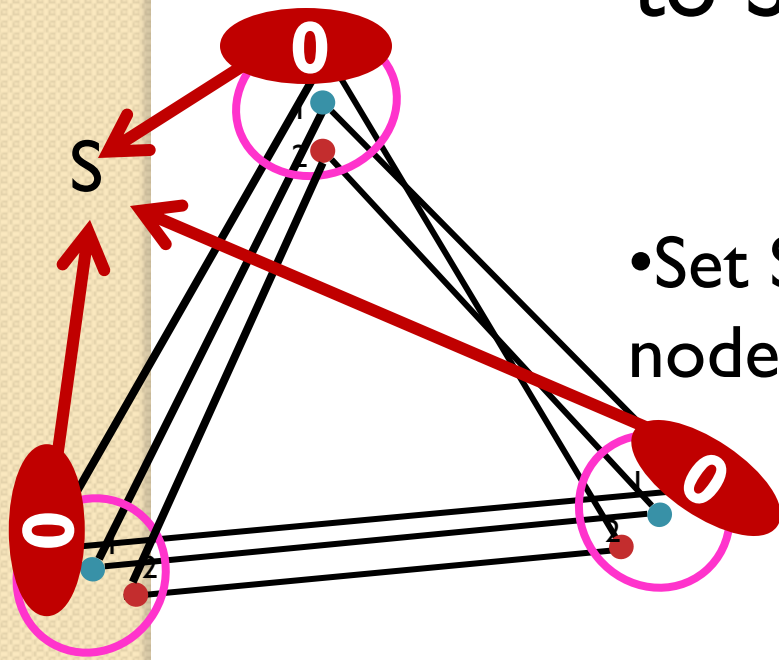


...unless UG instance has **expansion!** [KV,manuscript]

Because then any sparse cut would correspond to a good labeling

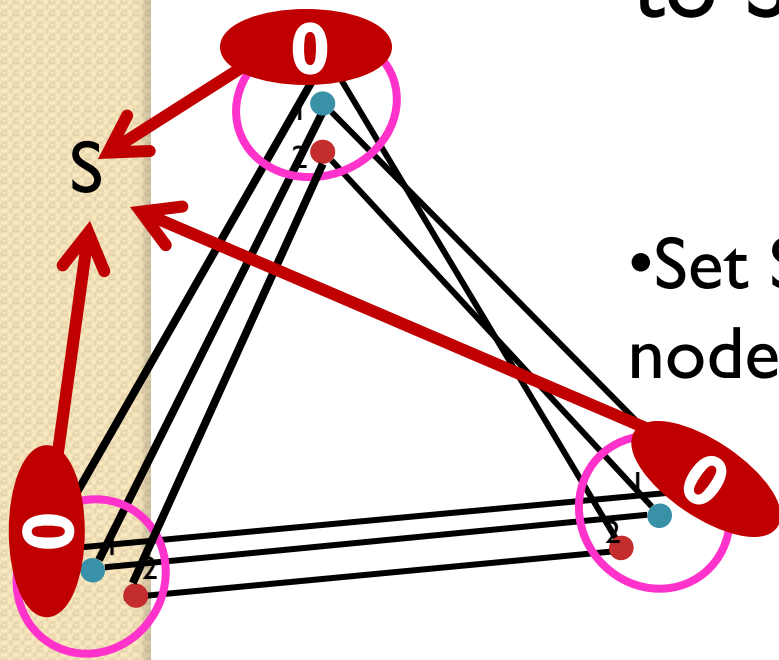
Off-the-record belief that **expanders** were hardest instances

Proof with Graph Theory: From Labelings to Spectra



- Set S that contains **exactly one** “small” node from each node group = **labeling**

Proof with Graph Theory: From Labelings to Spectra



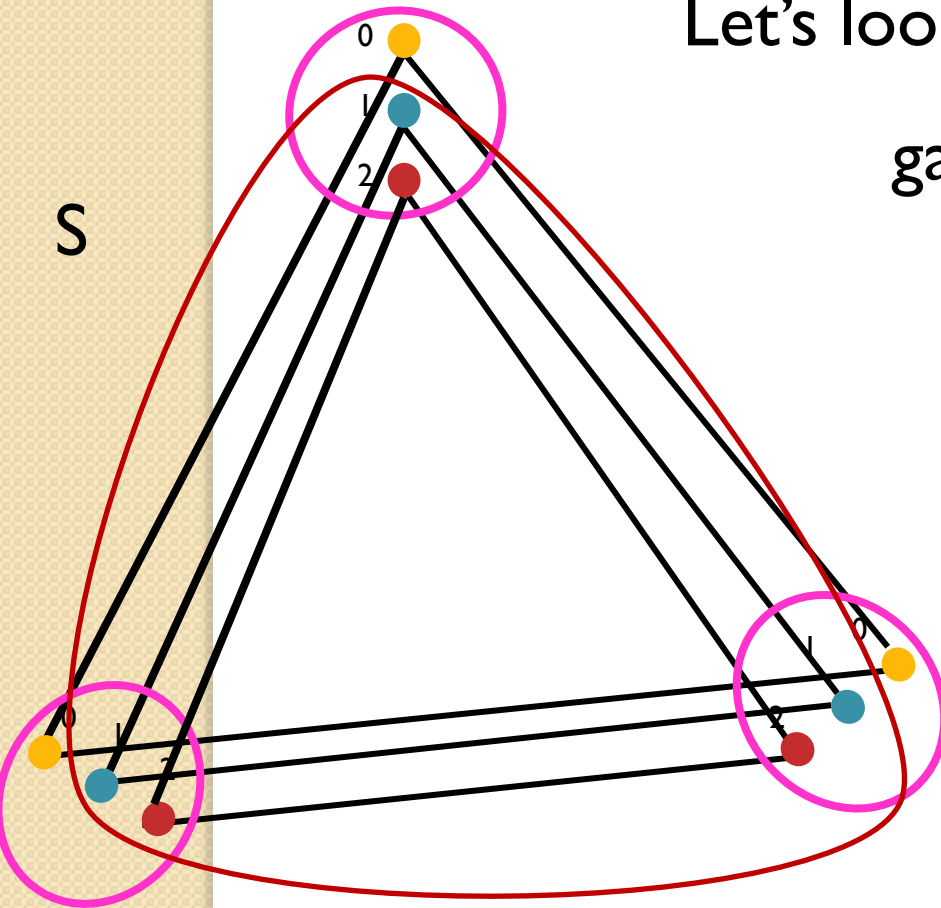
•Set S that contains **exactly one** “small” node from each node group = **labeling**

- Corresponds to a **cut** (S, S') .
- Corresponds to a “**characteristic vector**”.

$$\chi_{(0,0,0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Proof Intuition: a Perfect Game

Let's look at a **perfectly satisfiable** game for intuition...



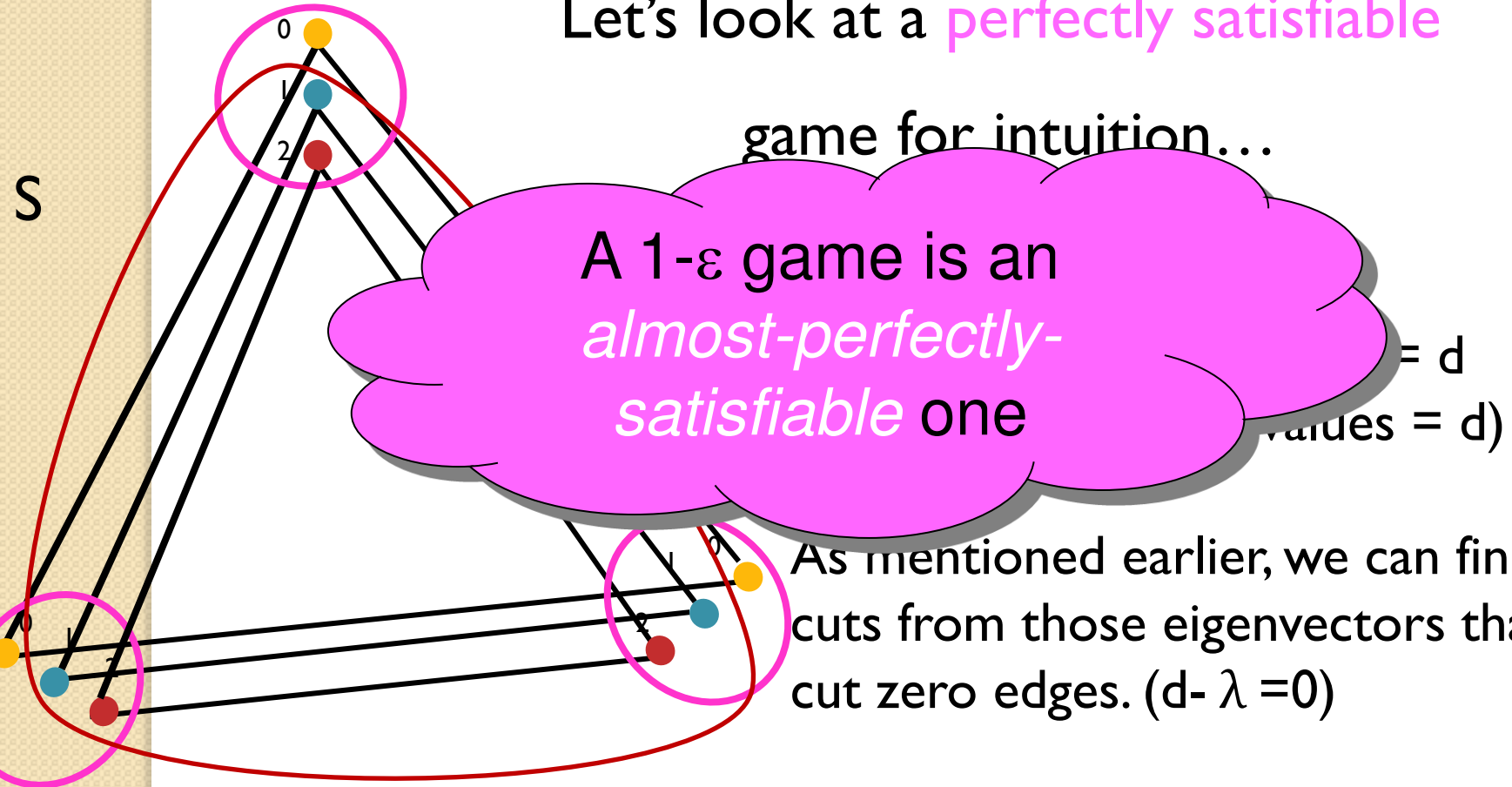
Graph is disconnected,
it has second eigenvalue $\lambda = d$
(in fact, it has k eigenvalues $= d$)

As mentioned earlier, we can find
cuts from those eigenvectors that
cut zero edges. ($d - \lambda = 0$)

If graph G was originally connected,
those are the only “sparsest cuts”.
They correspond to perfect labelings.

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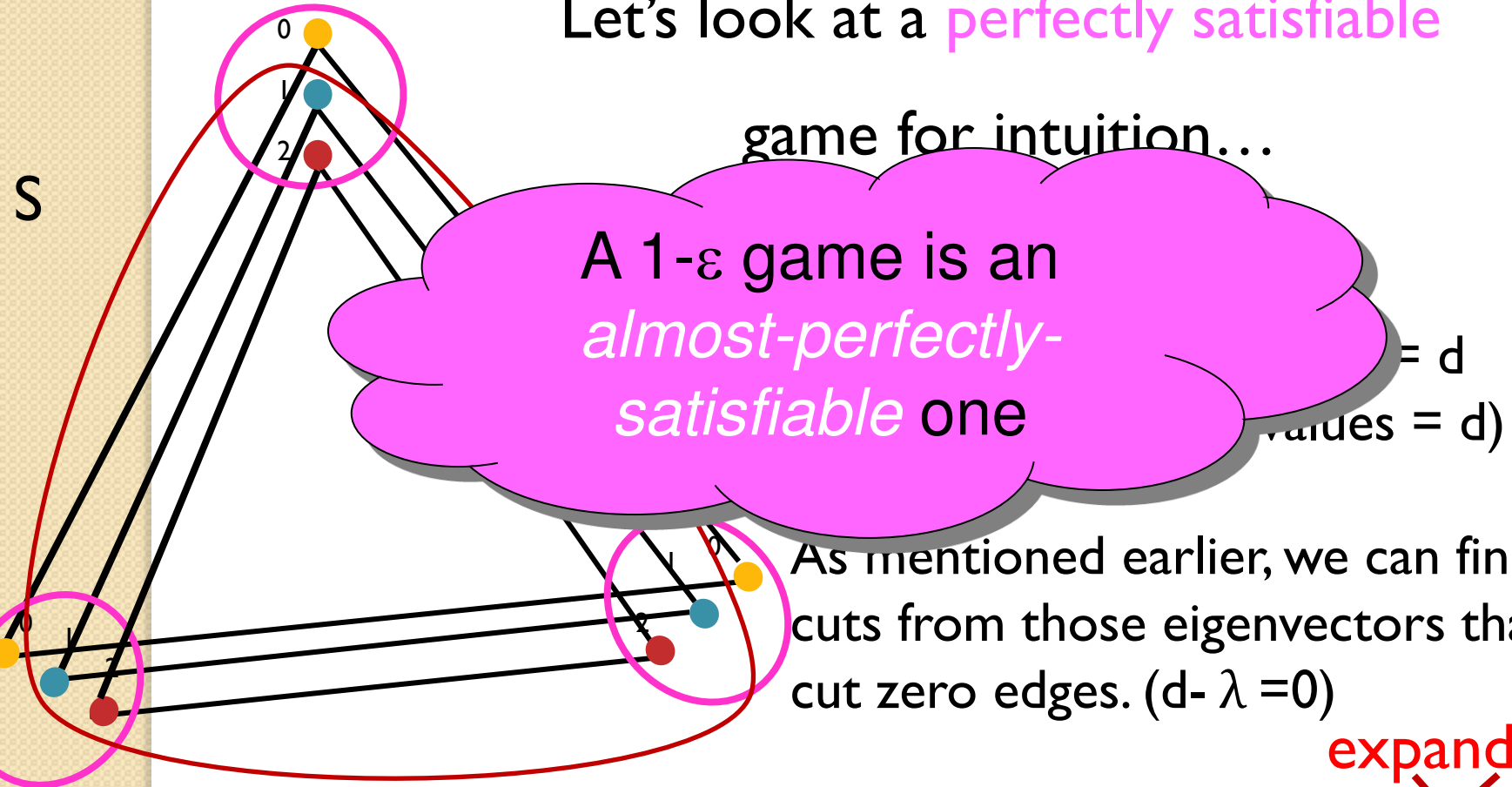
A $1-\epsilon$ game is an *almost-perfectly-satisfiable* one

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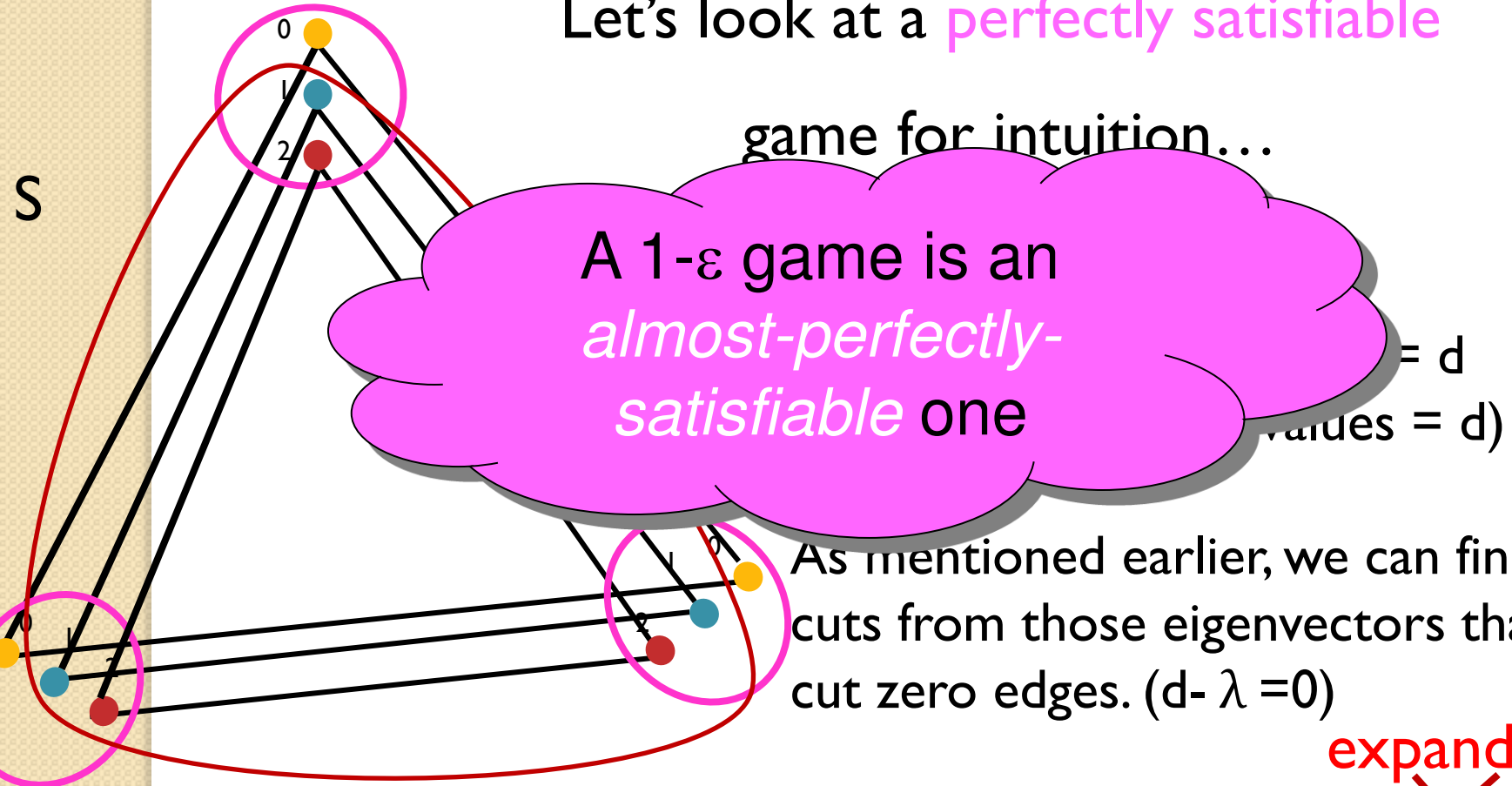
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expander

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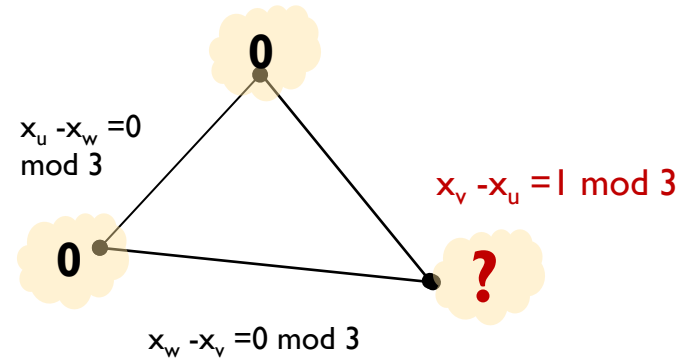
~~expander~~

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They correspond to **almost-perfect** labelings

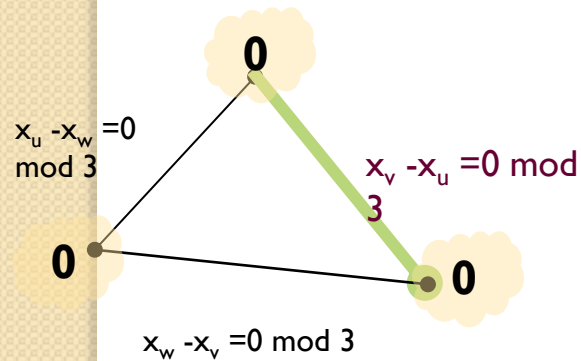
Proof: Reverse Engineering + Graph Spectra

I- ϵ Game



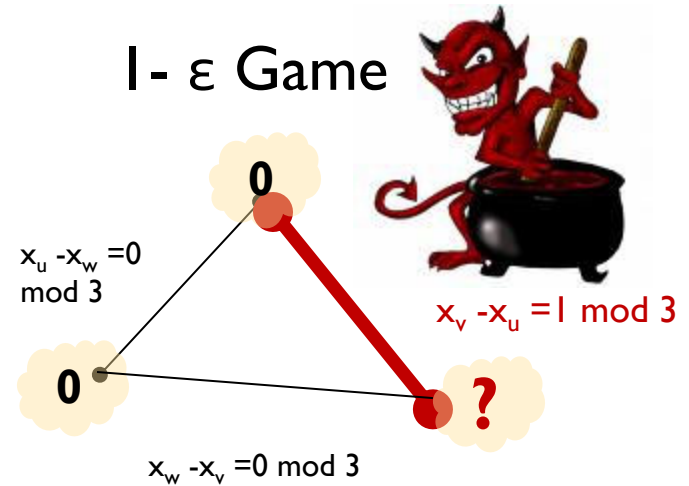
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Perfect Game:



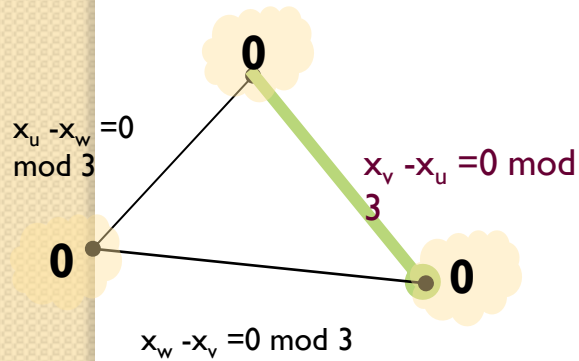
Think of it as “coming from” **adversarialy** perturbed completely satisfiable game

I-ε Game

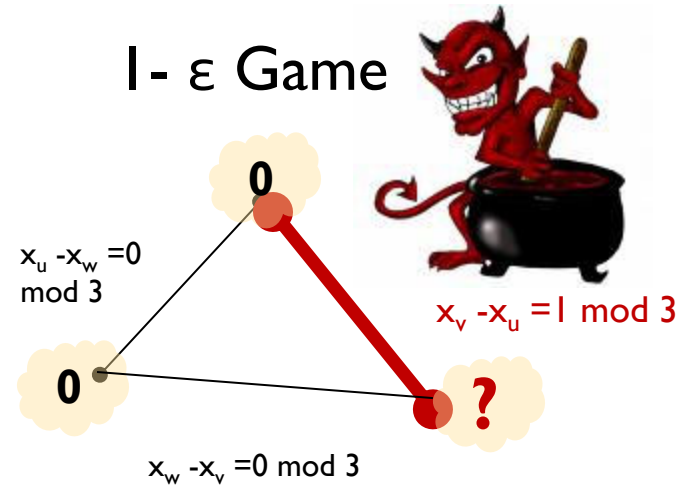


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Perfect Game:

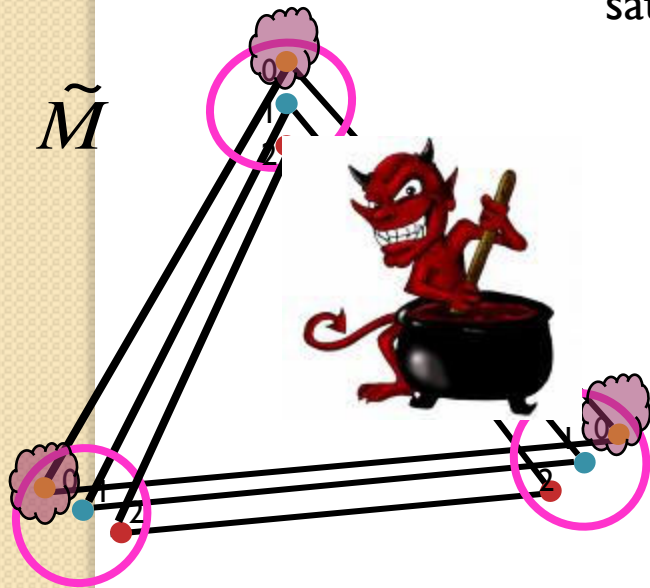


1- ϵ Game

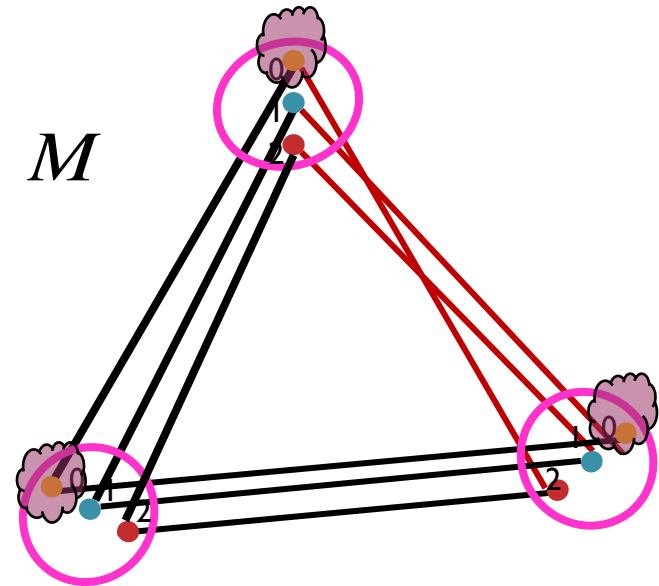


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\tilde{M}



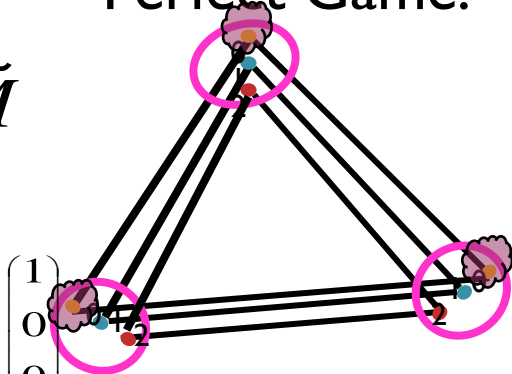
M



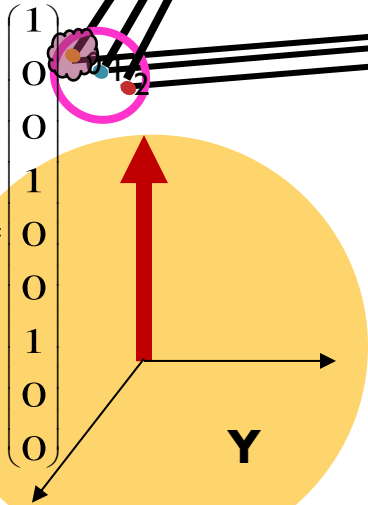
Proof: Reverse Engineering + Graph Spectra

Perfect Game:

\tilde{M}

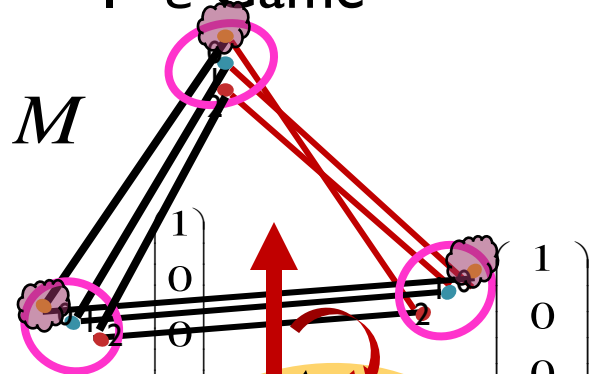


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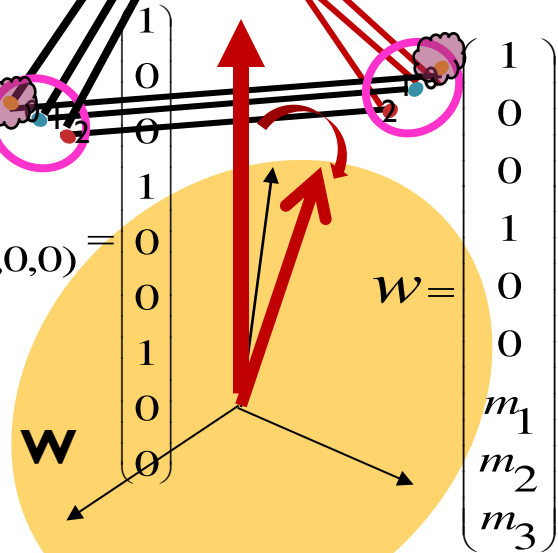


1-ε Game

M



$\chi_{(0,0,0)} =$



**First few
eigenvectors:**

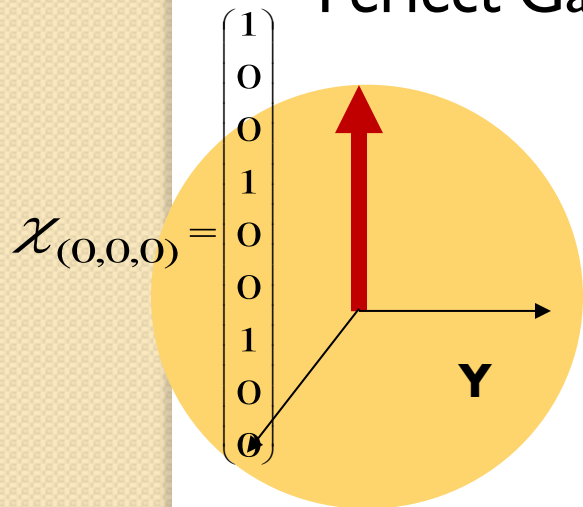
The k “labeling vectors” have large projection onto space W with values $> (1 - 200\epsilon)d$

“Labeling” eigenvectors:

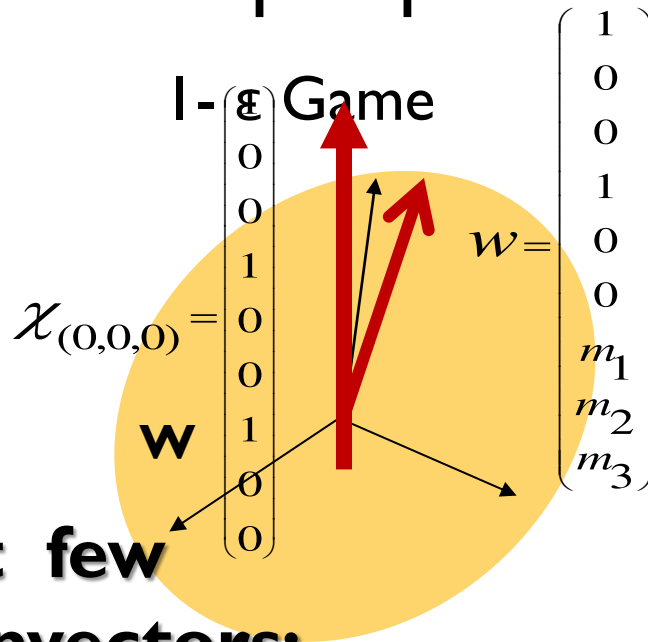
The k-dimensional space Y of values equal to d contains all the information for the best labeling

Proof: Reverse Engineering + Graph Spectra

Perfect Game:



$1 - \epsilon$ Game



First few eigenvectors:

“Labeling” eigenvectors:

The k -dimensional space Y of values equal to d contains all the information for the best labeling

The k “labeling vectors” have large projection onto space W with values $> (1 - 200\epsilon)d$

for $\|\chi\|=1, \chi^T \tilde{M} \chi = d$

$\chi^T M \chi \geq (1 - 2\epsilon)d$

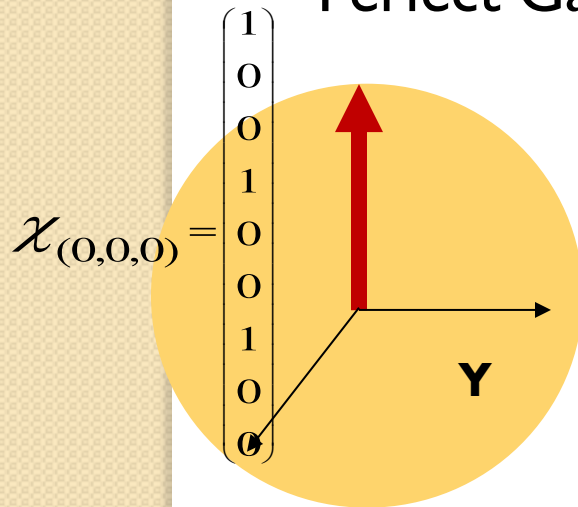
$(1 - 2\epsilon)d \leq \chi^T M \chi = \alpha^2 w^T M w + \beta^2 w_{\perp}^T M w_{\perp}$

Write: $\chi = \alpha w + \beta w_{\perp}$

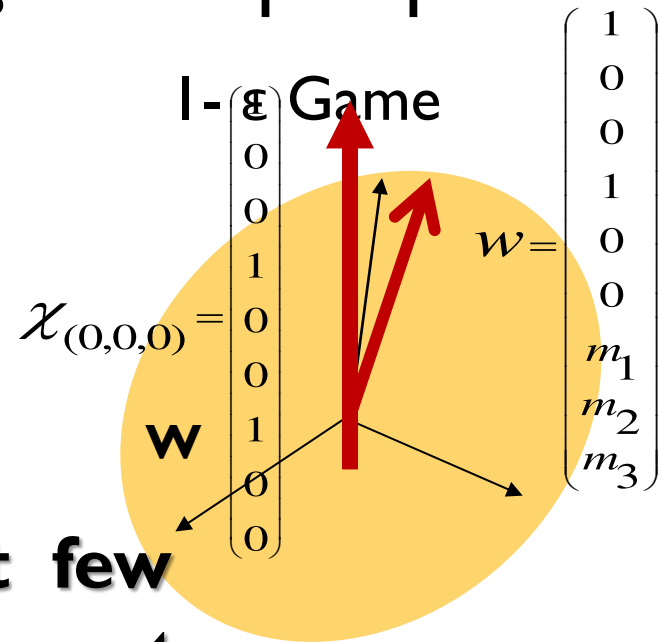
$\leq \alpha^2 d + \beta^2 (1 - 200\epsilon)d \Rightarrow |\beta| \leq \frac{1}{10}$

Proof: Reverse Engineering + Graph Spectra

Perfect Game:



$1 - \epsilon$ Game



First few
eigenvectors:

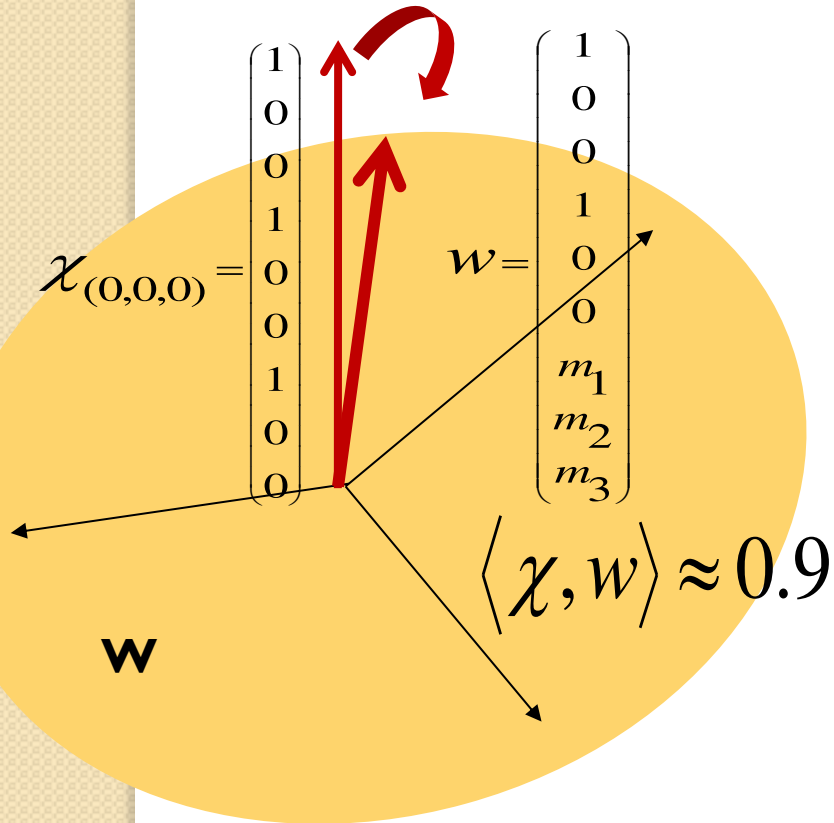
“Labeling” eigenvectors:

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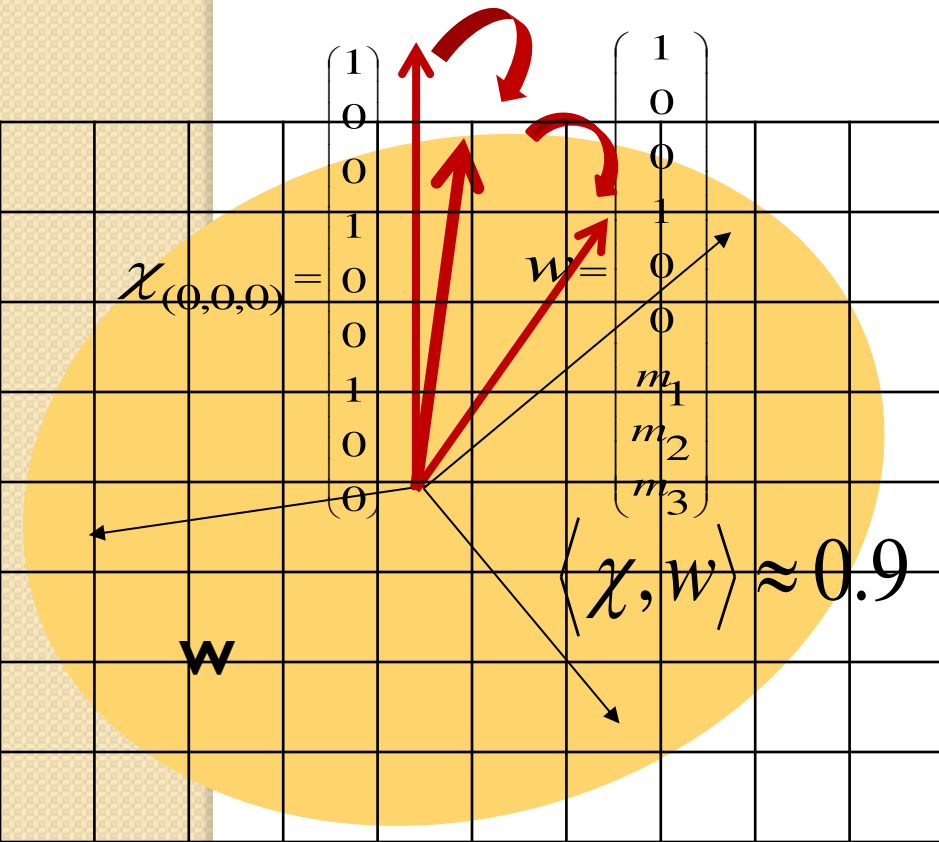
If we knew the projection w of χ then we could just “read off” a good labeling

Searching for a Needle in a Haystack?



But we need to find a particular vector in this whole space W !

Searching for a Needle, but “Efficiently”



But we need to find a particular vector in this whole space W !

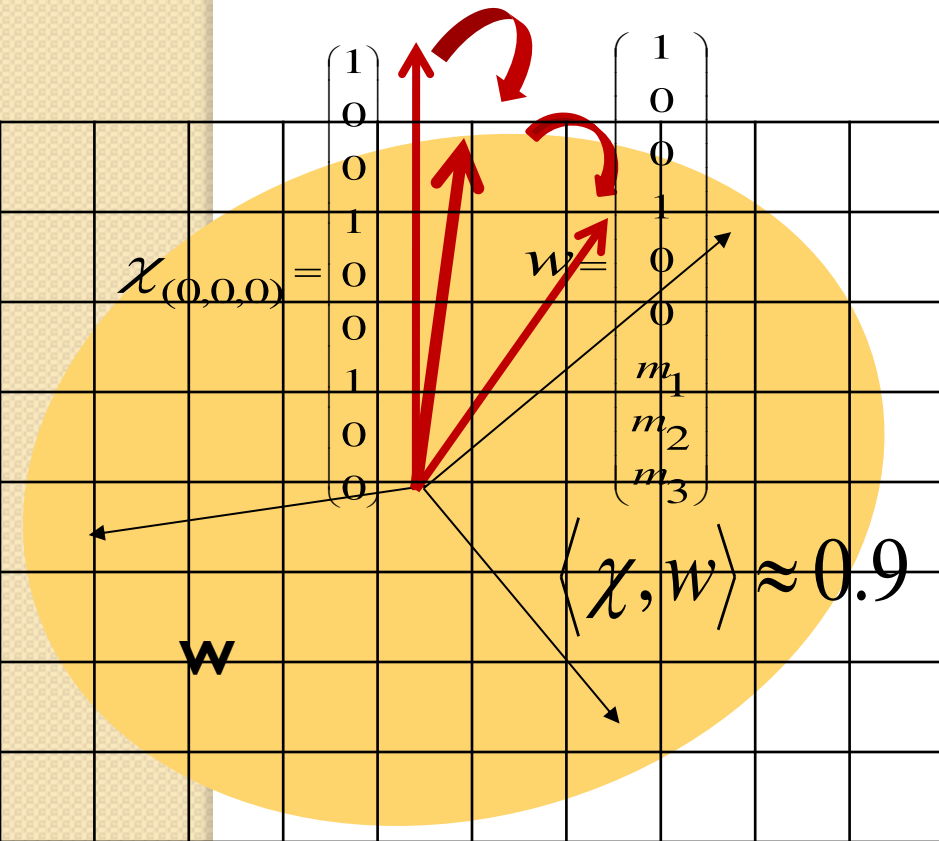
Idea:
Discretize the space by net!

One point of the net is close to the vector we want

We find this vector and then “read off” the coordinates

Most blocks have (unique) maximum entry in the position that corresponds to the original value of node u

Searching for a Needle, but “Efficiently”



But we need to find a particular vector in this whole space W !

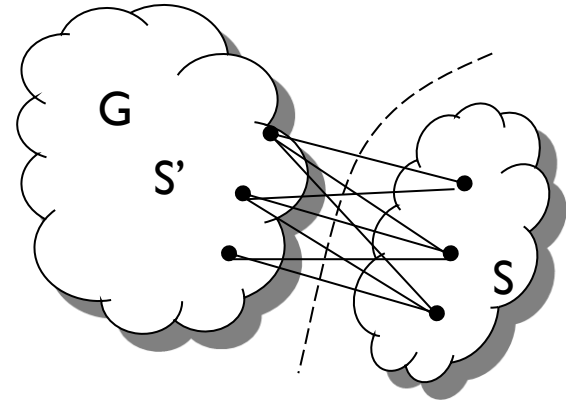
Idea:
Discretize the space by net!

Algorithm runs in time \sim #points in the net
=
exponential in the dimension of eigenspace W

The Dimension of W for Expanders

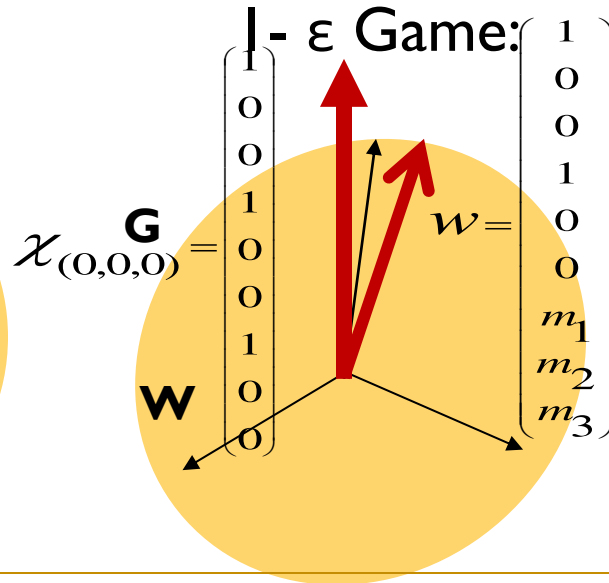
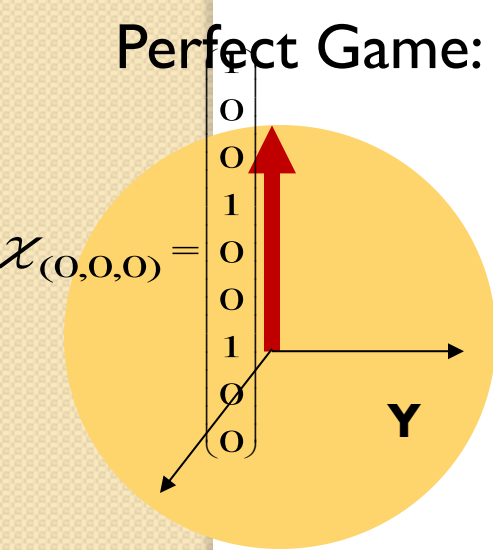
(Spectral Gap) =

$$d - \lambda = \gamma d$$



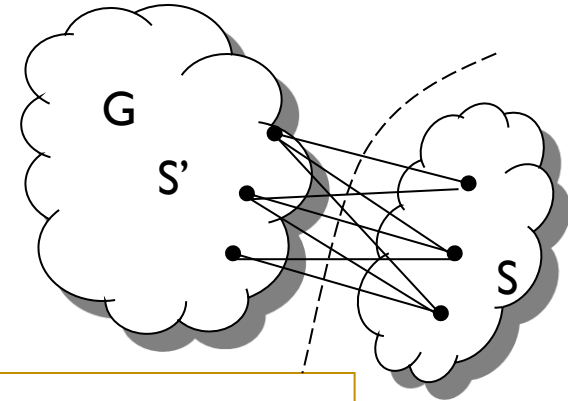
The Dimension of W for Expanders

Perfect Game:



(Spectral Gap) =

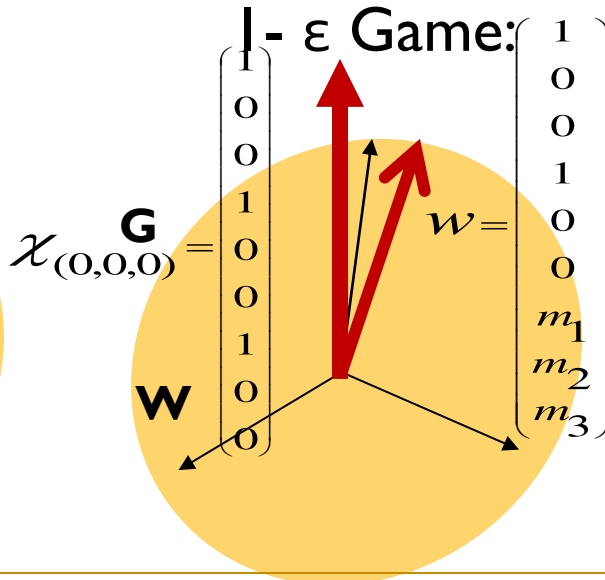
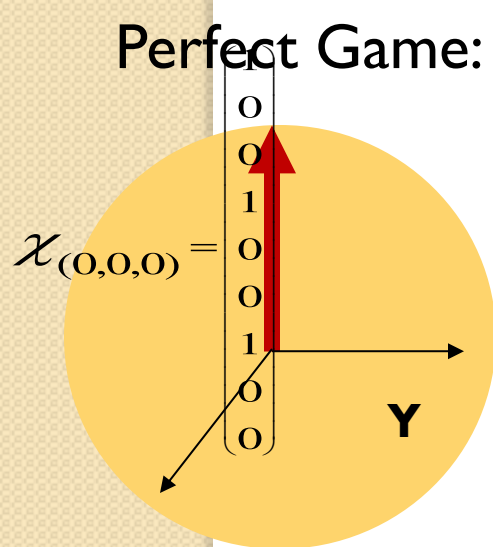
$$d - \lambda = \gamma d$$



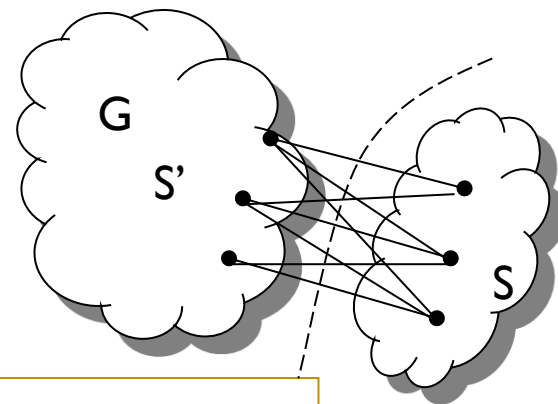
(Spectral gap between Y, Y_{\perp}) = absgap = γd

The Dimension of W for Expanders

Perfect Game:



(Spectral Gap) = $d - \lambda = \gamma d$



(Spectral gap between Y, Y_{\perp}) = $absgap = \gamma d$

W is “perturbed analog” of Y

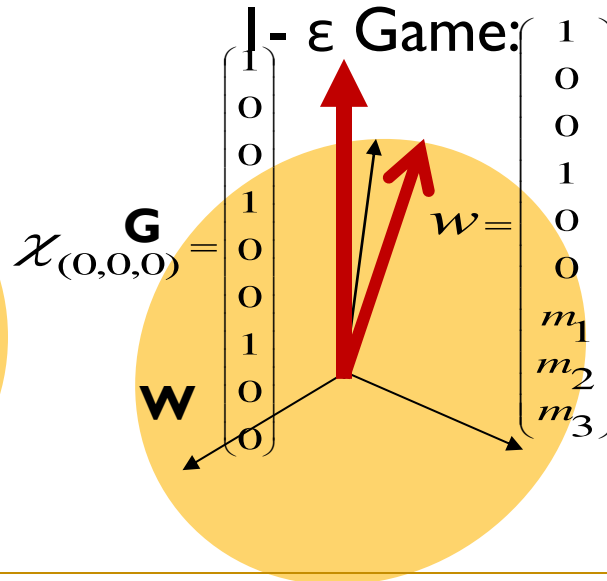
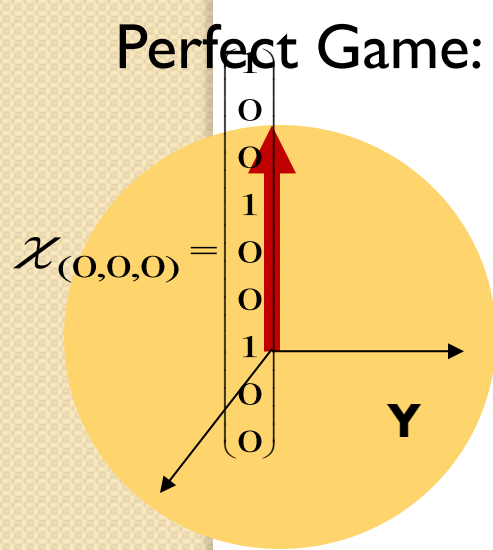
“The $\sin \mu$ ” Theorem [DK’70] Angle between Y and “perturbed analog of Y ” small

Equivalently, we can write every vector w in W as $w = \alpha y + \beta y_{\perp}, y$ in Y

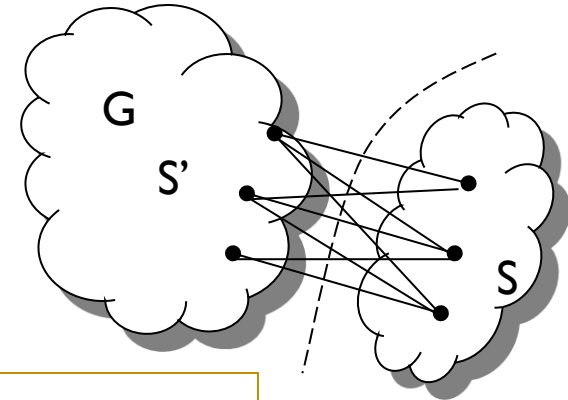
$$|\beta| \leq \frac{\|(M - M_{\epsilon})w\|}{absgap} \leq O\left(\sqrt{\frac{\epsilon}{\gamma^3}}\right)$$

The Dimension of W for Expanders

Perfect Game:



(Spectral Gap) = $d - \lambda = \gamma d$

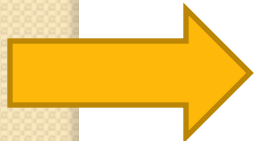


(Spectral gap between Y, Y_{\perp}) = $\text{absgap} =$

γd

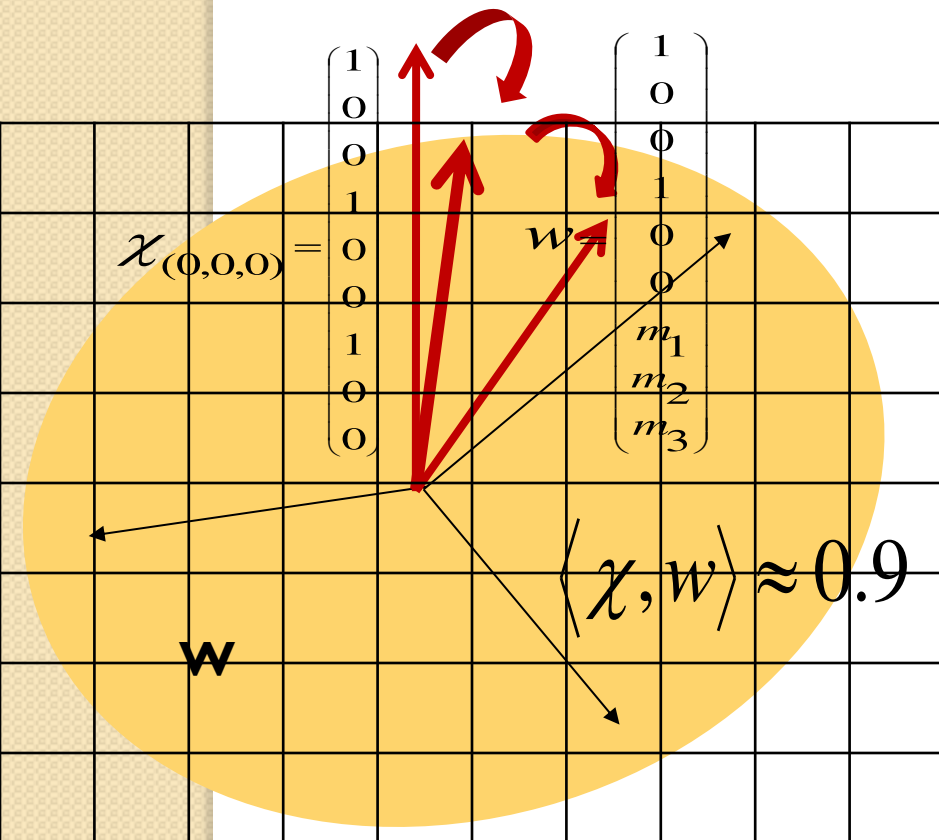
W is “perturbed analog” of Y

“The $\sin \mu$ ” Theorem [DK’70] Angle between Y and “perturbed analog of Y ” small



W is close to Y so $\dim(W) \leq \dim(Y) = k$

A General Algorithm

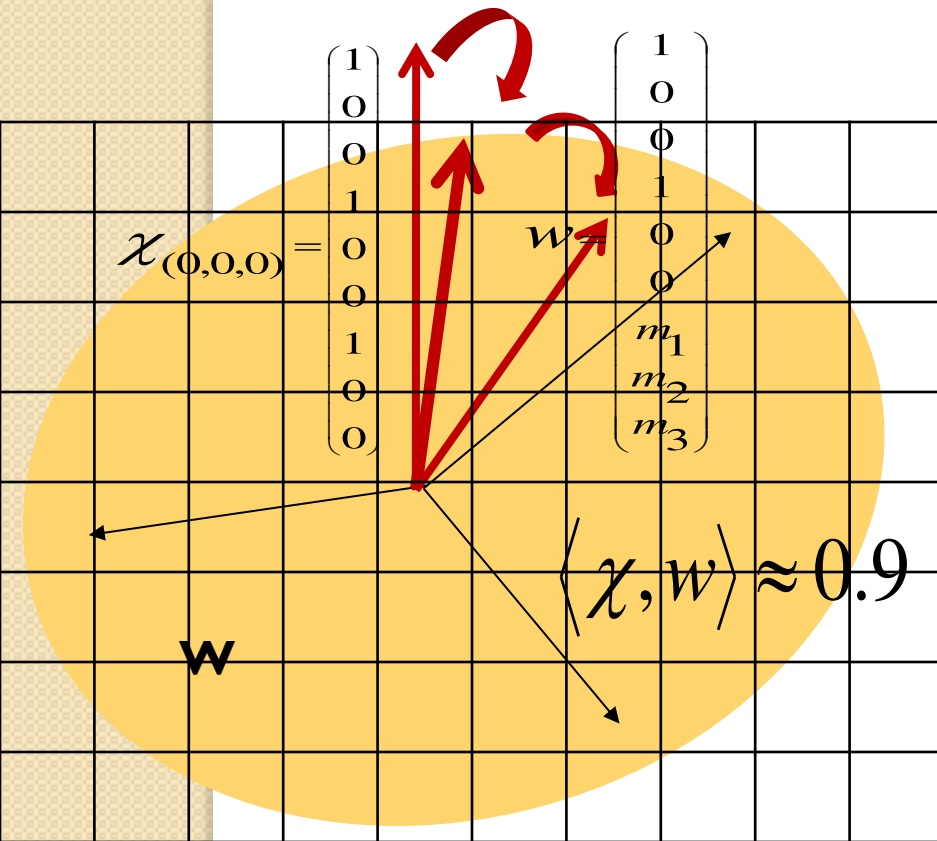


For expanders,
 W is close to Y so
 $\dim(W) \leq \dim(Y) = k$

Running time is
 $2^k \approx 2^{\log n} \approx \text{poly}(n)$

Algorithm runs in time \sim #points in the net
 $=$
exponential in the dimension of eigenspace W

A General Algorithm

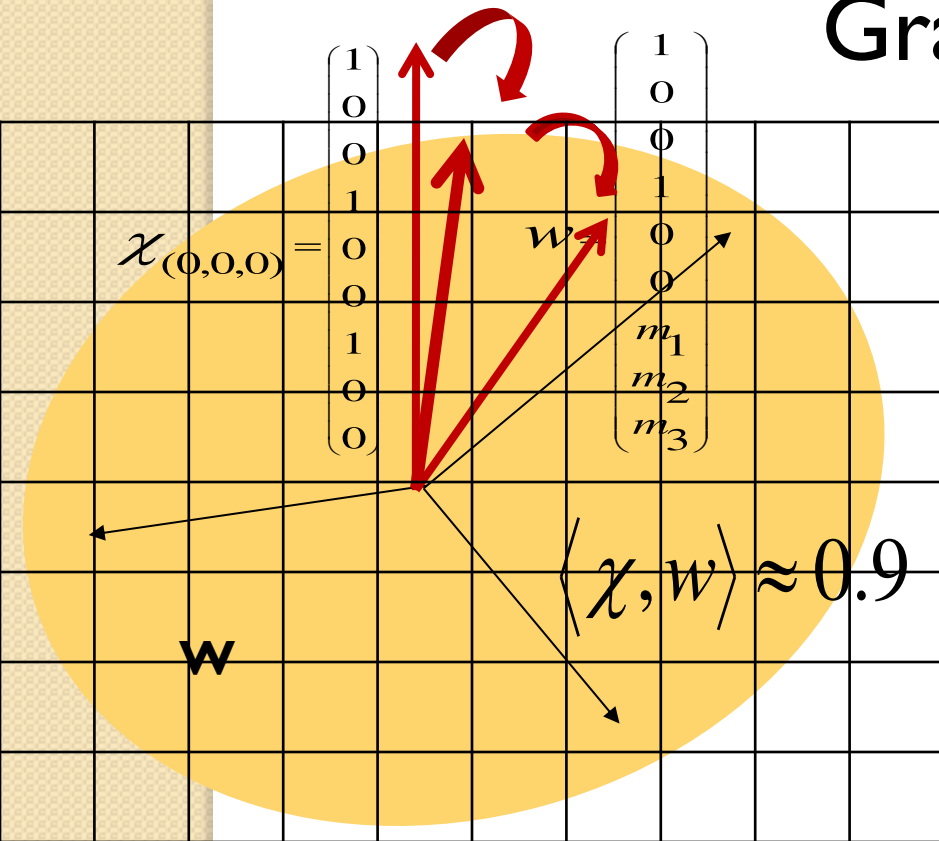


Algorithm runs in time \sim #points in the net

=

exponential in the dimension of eigenspace W

Another Special Case: The “Khot-Vishnoi” Graph

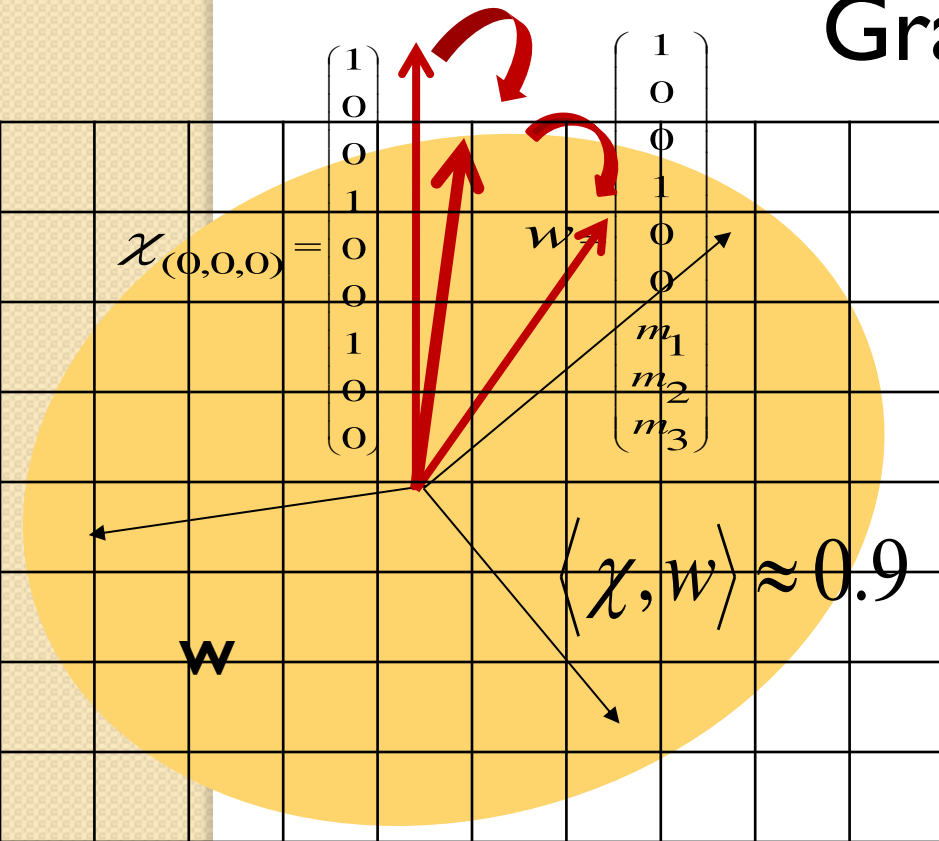


Graph that “cheats” a canonical semidefinite program for UG

We show: Eigenspace in question has poly-logarithmic dimension

Algorithm runs in time \sim #points in the net
 $=$
exponential in the dimension of eigenspace

Another Special Case: The “Khot-Vishnoi” Graph



Graph that “cheats” a canonical semidefinite program for UG

We show: Eigenspace in question has poly-logarithmic dimension

Algorithm runs in time \sim #points in the net

=

quasi-polynomial

UGC and the Spectrum of General Graphs

- After expanders, we realized that other constraint graphs are easy for UGC.
- How “easy” the graph is, depends on the number of large (close to d) eigenvalues of the adjacency matrix of the label-extended graph.
- Could solve previously “hardest” cases, where all Other techniques failed.
- Essentially only one case left, reflected by the Boolean Hypercube!! (?)

Open Questions

Disprove the Unique Games Conjecture

- Can we argue about UGC on the cube?
- About 2 years ago a group of Quantum Computing Theorists came together and tried to find a quantum algorithm...
- Proved Maximal Inequality on the Cube, failed for UGC.
- What is the quantum complexity of UGC?

THANK YOU!