CS 598: Spectral Graph Theory. Lecture 13

Eigenvalues of Random Graphs (or, Random Graphs are Expanders)

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Today

- Define random graph model.
- Concentration inequalities
- Bounding Raleigh Quotients
- Explain Trace method

Erdos-Renyi model

- Set n of vertices.
- Between every 2 vertices (u, v) add edge with probability p: $G(n, p)$.
- For this lecture $p = \frac{1}{2}$.
- Adjacency matrix has entries $A(i, \mathrm{j}) = 0$ with probability ½ and $A(i, \mathrm{j}) = 1$ with probability ½ .

The Expectation

- Let $M = \mathrm{E(A)} = \frac{1}{2} A_{K_n} = \frac{1}{2}(J_n - I_n)$ the expectation of the adjacency matrix
- One eigenvalue is (n-1)/2 and the others are -1/2.
- We will show that $||A - M|| \leq 1.34\sqrt{n}$ with exponentially high probability.
- $||A - M|| = \max_i |\lambda_i(A - M)| = \max_{\mathrm{x}} |\mathrm{x}^{\mathrm{T}}\mathrm{Rx}/\mathrm{x}^{\mathrm{T}}\mathrm{x}|$, where R=A-M.

One Raleigh Quotient

- R(i,j) is mean-zero r.v taking values $\pm 1/2$.
- Fix any unit vector x:
  $x^T Rx = \sum_{i<j} 2r_{ij} x(i)x(j)$.
- This is sum of independent r.v, with mean zero.
- **Hoeffding bound** for i.i.d $X_i, |X_i| \in [a_i, b_i], E(X_i) = \mu$:

$$\Pr[\sum_{i=1 \; to \; d} X_i \geq t + \mu] \leq e^{-2t^2/\sum_i (b_i - a_i)^2}$$

One Raleigh Quotient

- **Lemma.** For every unit vector x

$$\Pr_R[|x^T Rx| \geq t] \leq 2e^{-t^2}$$

- Kind of useless, there are infinitely many x's!!!

One Raleigh Quotient

- **Lemma.** Let R be a symmetric matrix and let u be a unit eigenvector of R whose eigenvalue has absolute value $||R||$. If x is another unit vector such that $\langle v, x \rangle \geq \sqrt{3}/2$, then $x^T R x \geq \frac{1}{2}||R||$.

How many vectors are close?

- **Lemma.** Let v be an arbitrary unit vector and x a random unit vector. Then
$$\Pr\left[\langle x, v\rangle \geq \frac{\sqrt{3}}{2}\right] \geq \frac{1}{\sqrt{\pi} n 2^{n-1}}$$

The Probabilistic Argument

- **Theorem.** Let R be a symmetric matrix as above.

$$\Pr[||R|| \geq t] \leq \sqrt{\pi}\mathrm{n}2^{\mathrm{n}}e^{-t^2/4}$$

An Alternative Argument

- Trace method used to bound eigenvalues of matrices. More powerful sometimes (e.g. eigenvalues of d-regular graphs)
- $Tr\left(A^k\right) = \sum_i \lambda_i^k = \sum k - walks(i,i)$
- $||R|| \leq \left(Tr\left(R^k\right)\right)^{1/k}$