CS 598: Spectral Graph Theory. Lecture 12

A construction of Expanders

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Today

- Determnistic construction of expanders
- Start from small graph, what to do next?
- Squaring the graph
- Replacement product
- A construction of 1/10- expanders.

Operations on a Graph

- We saw last time that we can start form a small good expander (can find one easily), perform some operation and get a bigger expander. Few ways to do it:
  - Lifts (in two weeks)
  - Squaring
  - Replacement Products
  - Combinations of the above

Attempt 1: Squaring a Graph

- We can improve the expansion of a graph by squaring it.
- So we could start from large non-expanding graph and get expander perhaps? ☺

- **Definition**: We define the square of a graph G^2 to be a the graph with adjacency matrix $A_{G^2} = A_G^2 - dI_n$

- G^2 has an edge between (u,v) if (u,v) are connected with path of length two. Remove self-loops.
- If graph has no 4-cycles, then $A_G^2(u, v) = 1$ for such two nodes.

Squaring a Graph

- **Theorem**:

Let G be a d-regular graph with Laplacian eigenvalues $\lambda_1, \dots, \lambda_n$. Then, G^2 is a d(d-1) regular graph with Laplacian eigenvlaues $2d\lambda_i - \lambda_i^2$. In particular, the largest eigenvalue is at most d^2.

- If G is ϵ-expander, then ratio of eigenvalues to degree is $(1 \pm \epsilon)$.
- Squaring the graph, we get ratio of eigenvalues to degree $(1 \pm \epsilon^2)$, so it is an ϵ^2 - expander.

Squaring a Weak Expander

- By previous theorem we can see that if a graph is a weak expander in the sense $\lambda_2 = \delta d$, where $\delta \ll 1$, then the expansion almost doubles by squaring.

- The degree also squares –not so useful ☹

Attempt 2: Line Graph

- Line graph H of G has a vertex per edge of G, where two are connected if they share an endpoint in G.
- G d-regular has $nd/2$ edges, so H has that many vertices.
- H has degree 2(d-1)
- If we consider one vertex u of G, then all the d edges (u,v) will be connected in H: They form a d-clique!
- Every u belongs to two d-cliques.

Spectrum of Line Graph

- **Theorem.**

Let G be a d-regular graph with n vertices and H be the line graph. Then the spectrum of the Laplacian of H is the same as the spectrum of the Laplacian of G, except that it has $\frac{nd}{2} - n$ extra eigenvalues equal to 2d.

Attempt 2: Line Graph

- Line graph is bigger than G, and degree is only a factor of two bigger, so we are in better shape, but still need to keep degree down ☹

- Measure the quality of expansion with spectral ratio: $r(G) = \min\left(\frac{\lambda_2}{d}, \frac{2d-\lambda_n}{d}\right)$
- If G is a-expander then r(G) >1-a.
- The closer to 1 (bigger) r(G) is, the better the expansion.

Spectrum of Line Graph

- **Theorem.**

Let G be a d-regular graph for d>5 and let H be its line graph. Then

$$r(H) = \frac{\lambda_2(G)}{2(d-1)} \geq \frac{r(G)}{2}$$

Plan of Attack

- We see that line graph has half the spectral radius of G. But is also has more vertices.
- Plan is to build an infinite family of d-regular graphs with spectral ratio bounded below by some absolute constant b:
- Begin with small expander.
- Take the line graph (Increase size)
- Replace the cliques with expanders (decrease degree).
- Square (improve expansion).
- Repeat.

Approximation of Line Graph

- In order to increase degree, we approximate line graph.
- We know how to approximate cliques with expanders.
- Let's replace every d-clique in the line graph of G with an expander Z of d nodes and degree z call this new graph $Go_L Z$

Approximation of Line Graph

- **Theorem**. Let G be a d-regular graph. Let H be the line graph of G and let Z be a z-regular $\epsilon -$ expander. Then

$$(1-\epsilon)\frac{z}{d}H \preccurlyeq Go_L Z \preccurlyeq (1+\epsilon)\frac{z}{d}H$$

Approximation of Line Graph

- **Corrolary**.

$$r(G o_L Z) \geq \frac{(1-\epsilon)}{2} r(G)$$

- Now can we put all of those things together to get d-regular expanders?

Putting it Together

- Start from G_0 a d-regular graph, with spectral ratio $\beta = \frac{1}{5}$ (say). Also take Z to be a z-regular ϵ −expander on d nodes where $d = \left(2z(2z-1)\right)^2 - 2z(2z-1)$. Take $\epsilon = \frac{1}{6}$ (say).
- Goal is to create arbitrarily large d-regular graphs of same spectral ratio (or better).

Putting it Together

- Construct $G o_L Z$. Degree is 2z, spectral radius about $\beta/2$. Need to increase spectral radius (need it to be bigger than β for every size graph) , so square twice.
- Let $\mathrm{G}_1 = ((G o_L Z)^2)^2$
- Need Z of degree z and number of nodes d=$\left(2z(2z-1)\right)^2 - 2z(2z-1)$ as taken.
- Repeat, and set $\mathrm{G}_\mathrm{i} = ((G_{i-1} o_L Z)^2)^2$