CS 598: Spectral Graph Theory. Lecture 11

Cayley Graphs

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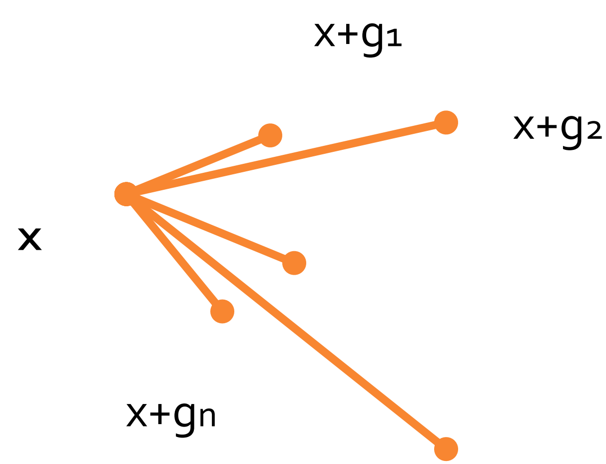
Today

- Groups
- Cayley graphs
- Eigenvectors and Eigenvalues of Cayley graphs of abelian groups
- Random Cayley graphs with logarithmic degree are expanders

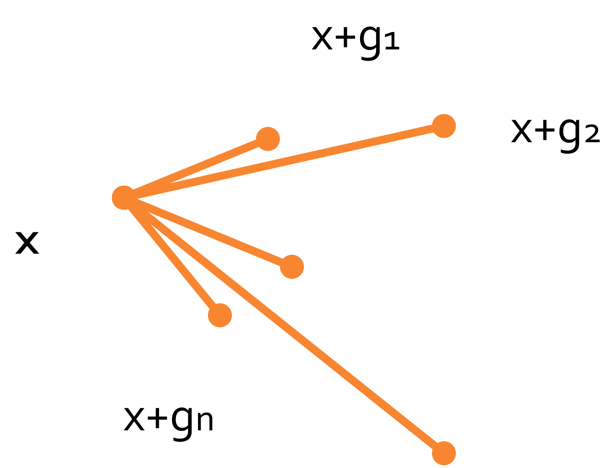
Graphs Like the Hypercube

- V= $\{0,1\}^m$ $(Z_2^m\ group)$
- $E = \{x, x + g_j, g_j \in \{0,1\}^m\ with\ at\ most\ t\ 1's\ \}$
- x,y neighbors if they differ in t coordinates. Degree d= $\sum_{\{i=1\ to\ t\}} \binom{m}{i}$.
- Cayley graph of Z_2^m $group$, if t=1 then its the Hypercube.



Graphs Like the Hypercube

- More generally,
- V= $\{0,1\}^m$ $(Z_2^m\ group)$
- $E = \{x, x+g_j, g_j \in S, for\ any\ set\ S\ of\ generators\}$



Evectors and Evalues

- For each $b \in \{0,1\}^m$, define the function $v_b: V \to R$,

$$v_b(x) = (-1)^{b^T x}$$

Think of b as being an index for Fourier coefficient

Evectors and Evalues

Theorem: For each $b \in \{0,1\}^m$, v_b is a Laplacian evector with evalue

$$\sum_{i=1\ to\ d} (1 - v_b(g_i)) = 2|Gb|$$

- Where G is a matrix with the g_i as rows.
- If S is set of strings with hamming weight 1, then $\lambda_2 = 2$ (hypercube).
- If S is all the strings, then complete graph.
- If S is linear code with min distance r, then $\lambda_2 = 2\mathrm{r}$. For expanders we need $d = O(\log n)$.

Groups

- Graphs constructed from groups are called Cayley graphs.
- Group is defined by set of elements, Γ and a binary operation $\circ$
- For elements g,h in Γ $g \circ h$ is also an element of Γ.

Groups

- (Γ , $\circ$) form a group if
  - Γ contains a special element called the identity (id) such that $g \circ id = id \circ g = g$ for all $g \in \Gamma$
  - For every element $g \in \Gamma$, there is another element $g^{-1} \in \Gamma$ such that $g^{-1} \circ g = g \circ g^{-1} = id$
  - For every three elements $f, g, h \in \Gamma$ $f \circ (g \circ h) = (f \circ g) \circ h$
  - Group is abelian if for every $g, h \in \Gamma$, *if*

    $g \circ h = h \circ g$

Groups: Examples You Already Know

1. (Integers , +)
2. (Z/n=Integers mod n, addition mod n)
3. (Z/p-0=Integers mod prime without zero, multiplication)
4. ($\{0,1\}^k$, componentwise addition mod 2) ; every element is its own inverse
5. For some k>0, (set of non-singular k-by-k matrices over integers, addition)
6. For some k>0, (set of non-singular k-by-k matrices over integers, multiplication)

Today: groups 2 and 4 : finite, abelian!

Cayley Graphs

- Cayley graph is defined by
  - a group $(\Gamma, \circ)$
  - a set of generators $S \subseteq \Gamma$ that is closed under inverse. That is, for every $g \in \Gamma, g^{-1} \in \Gamma$

  The vertex set of Cayley graph is Γ and the edges are the pairs

$$\{(g,h): h = g \circ s, some\ s \in S\} = \{(g, g \circ s): s \in S\}$$

E.g. Ring graph on n vertices if $(\Gamma, \circ) = (Z/n, +)$

And S=(-1,+1) (-1 =n-1 mod n)

Evectors and Evalues of Cayley Graphs of Abelian Groups

- We can find orthogonal set of eigenvectors without knowing S. Eigenvectors only depend on group
- Will consider adjacency matrix but doesn't really matter since Cayley graphs are regular.
- Next we re-prove the evectors of the "generalized" ring graph =(Z/n,+) and any set S of generators

Evectors and Evalues of Cayley Graphs of Abelian Groups

- For cycle, we showed that the eigenvectors are

$$x_k(u) = \sin\left(\frac{2\pi k u}{n}\right) \; and$$

$$y_k(u) = \cos\left(\frac{2\pi k u}{n}\right)$$

- Same eigenvectors if we consider any set of generators S. Eigenvalue $\sum_{g\in S} \cos(\frac{2\pi k g}{n})$

Expanders from Abelian Groups?

- We show that if we choose a d =clogn set of generators at random, we get a good expander w.h.p.
- Chernoff-Hoeffding Bound for i.i.d $X_i, |X_i| \leq 1, E(X_i) = 0$:

$$\Pr[\sum_{i=1\ to\ d} X_i \geq t] \leq e^{-t^2/4d}$$