CS 598: Spectral Graph Theory. Lecture 8

PRGs and Random Walks on Expanders

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Today

- Why PRGs?
- The use of PRGs in randomized algorithms
- Introduce expander graphs
- Random walks on expanders and Impagliazzo-Zuckerman PRG

Expander Graphs

- We will study expanders a lot the next few weeks.
- Constant degree (regular typically), constant conductance.
- By Cheeger we saw that we can characterize expanders through eigenvalues.
- A family of graphs is expanding if for i>1: $|\lambda_i - d| \leq \epsilon d$ or $|\mu_i| \leq \epsilon d$

Why Study PRGs?

- Pseudo-random number generators take a seed which is presumably random and generate a long string of random bits that are supposed to act random.
- Why would we want a PRG?
  - Random bits are scarce (eg low-order bits of temperature of the processor in computer is random, but not too many such random bits). Randomized algorithms often need many random bits.
  - Re-run an algorithm for debugging, convenient to use same set of random bits. Can only do that by re-running the PRG with the same seed, but not with truly random bits.

Why Study PRGs?

- Standard PRGs are terrible (e.g. rand in C). Often produce bits that behave much differently than truly random bits.

- One can use cryptography to produce such bits, but much slower

Repeating an Experiment

- Consider wanting to run the same randomized algorithm many times.
- Let A be the algorithm, which returns "yes"/"no" and is correct 99% of the time (correctness function of the random bits)
- Boost accuracy by running A t times and taking majority vote
- Use truly random bits the first time we run A and then with the PRG we will see that every new time we only need 9 random bits.
- If we run t times, probability that majority answer is wrong is exponential in t.

The Random Walk Generator

- Let r be the number of bits out algorithm needs for each run: space of random bits is $\{0,1\}^r$
- Let X$\subseteq \{0,1\}^r$ be the settings of random bits on which algorithm gives wrong answer for specific input.
- Let Y =$\{0,1\}^r \backslash X$ be the settings on which algorithm gives the correct answer

The Random Walk Generator: Expander Graphs

- Our PRG will use a random walk on a d-regular G with vertex set $\{0,1\}^r$, and degree d = constant.
- We want G to be an expander in the following sense: If A_G is G's adjacency matrix and $d = \alpha_1 > \alpha_2 \geq \cdots \geq \alpha_n$ its eigenvalues then we require that

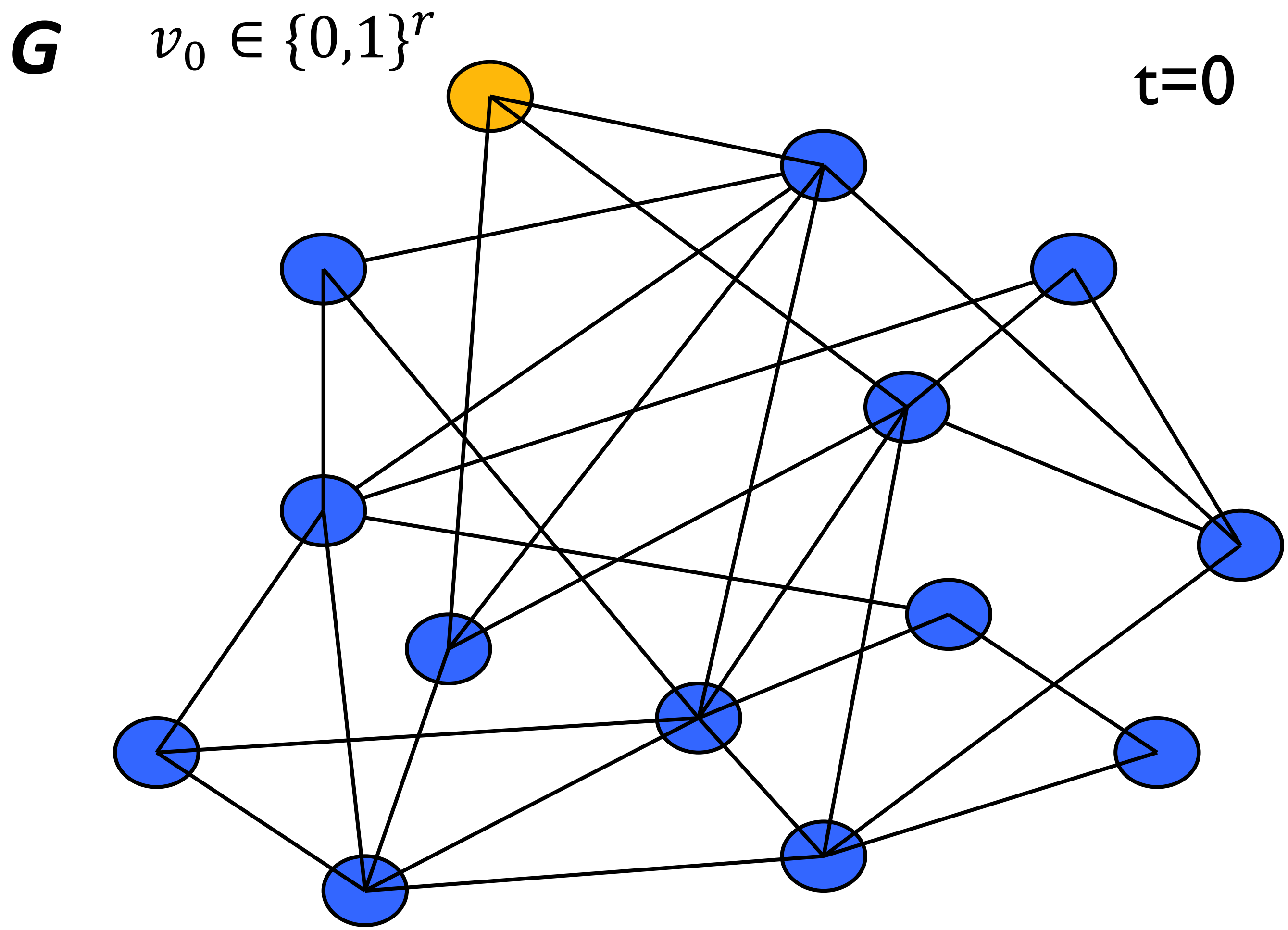
$$\frac{|\alpha_i|}{d} \leq \frac{1}{10}$$

Such graphs exist with d=400 (next weeks)

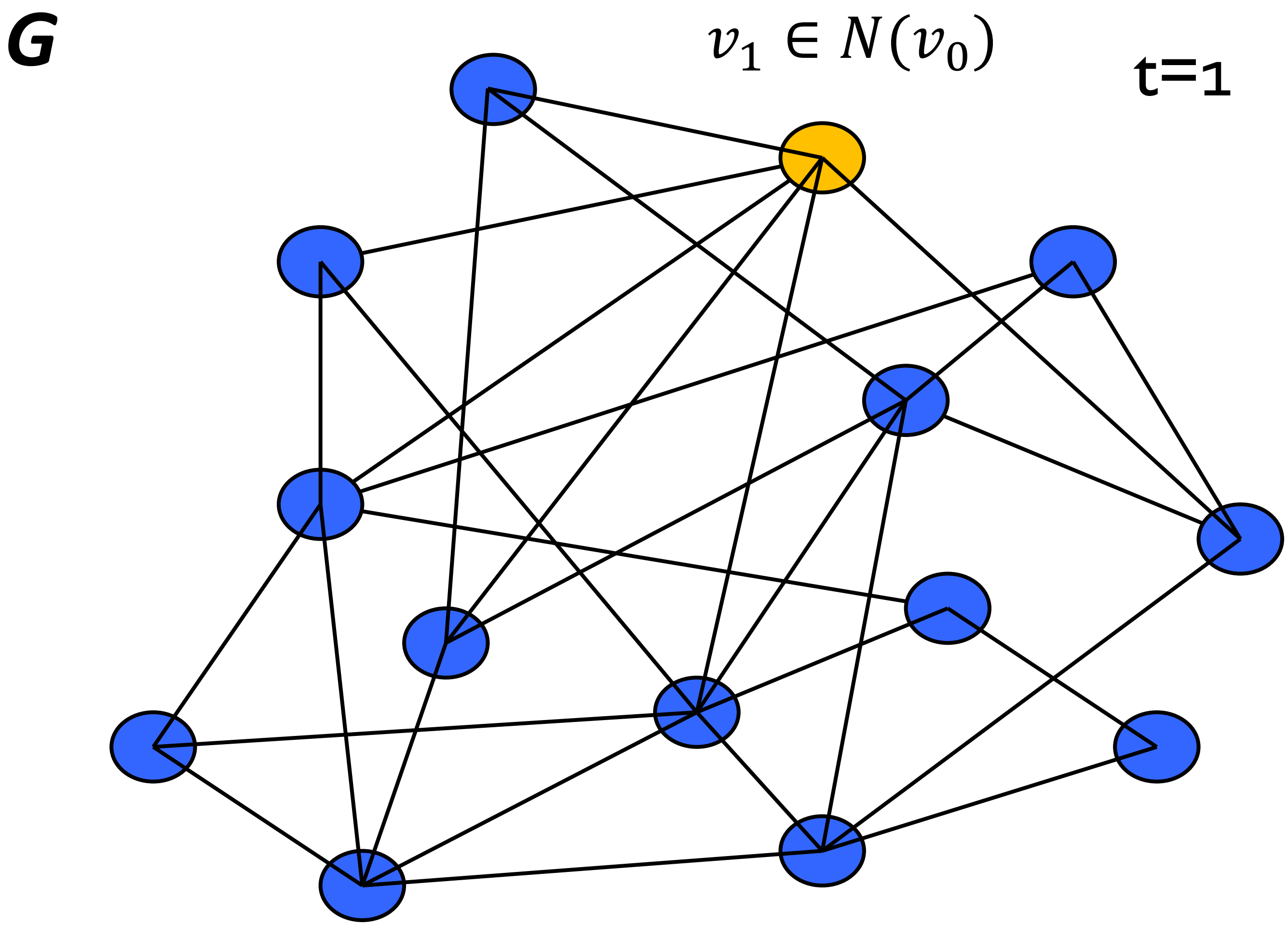
The Random Walk Generator

- For the first run of algorithm, we require r truly random bits. Treat those bits as vertex of expander G.
- For each successive run, we choose a random neighbor of the present vertex and feed the corresponding bits to our algorithm.
- I.e, choose random i between 1 and 400 and move to the i-th neighbor of present vertex. Need log(400) ~ 9 random bits.
- Need concise description, don't want to store the whole graph (e.g. see hypercube)

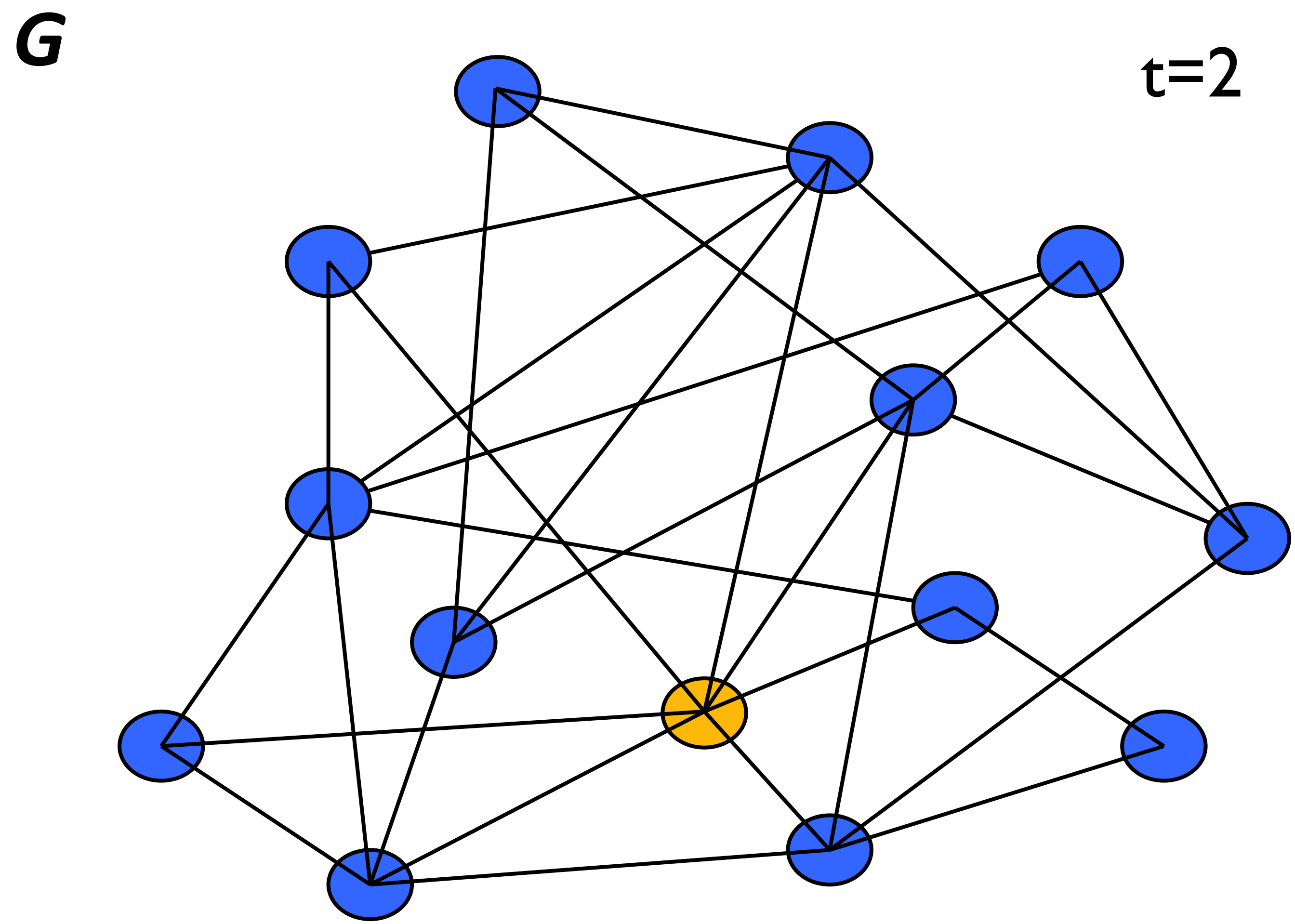
The Random Walk Generator



The Random Walk Generator

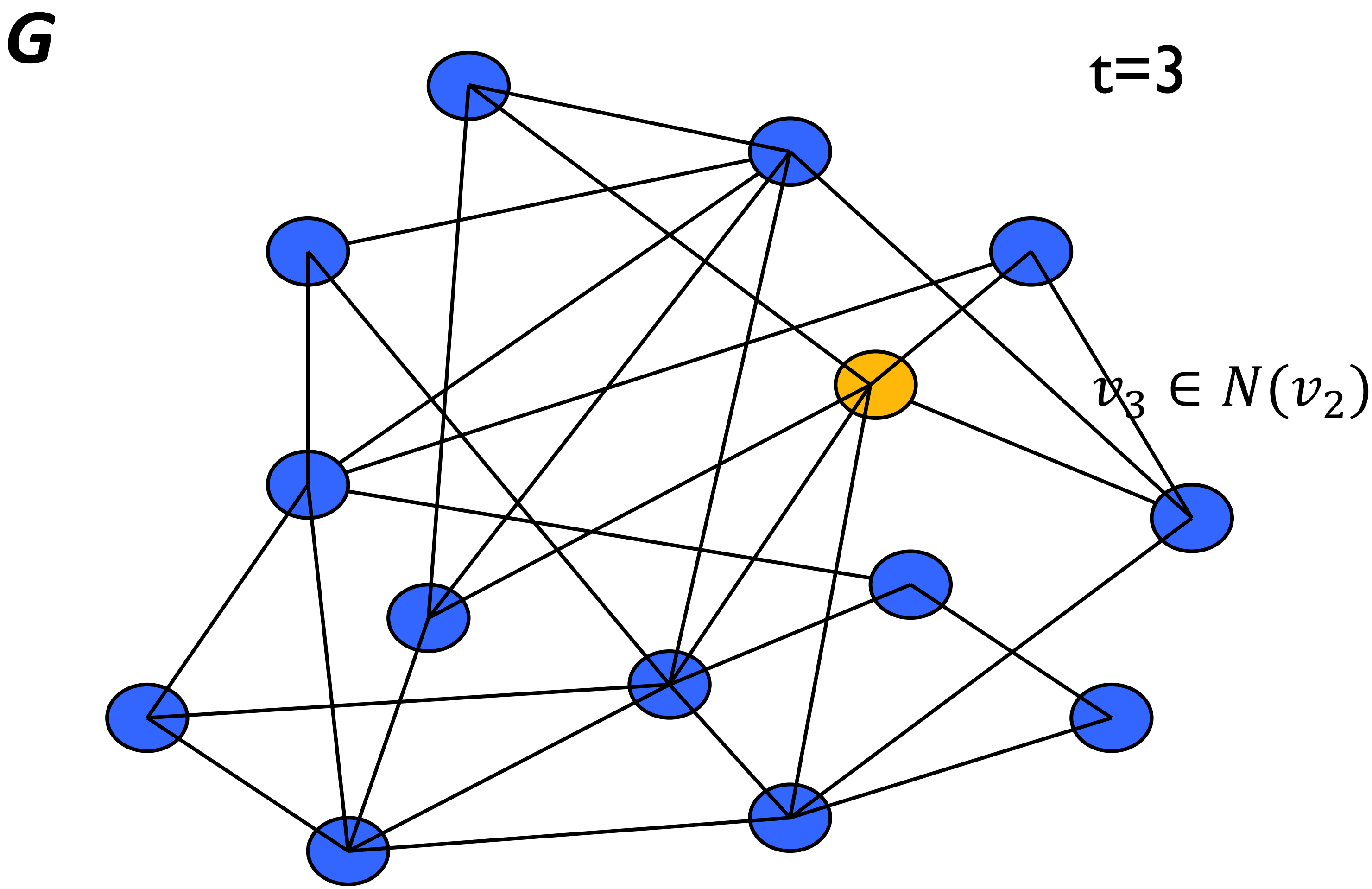


The Random Walk Generator

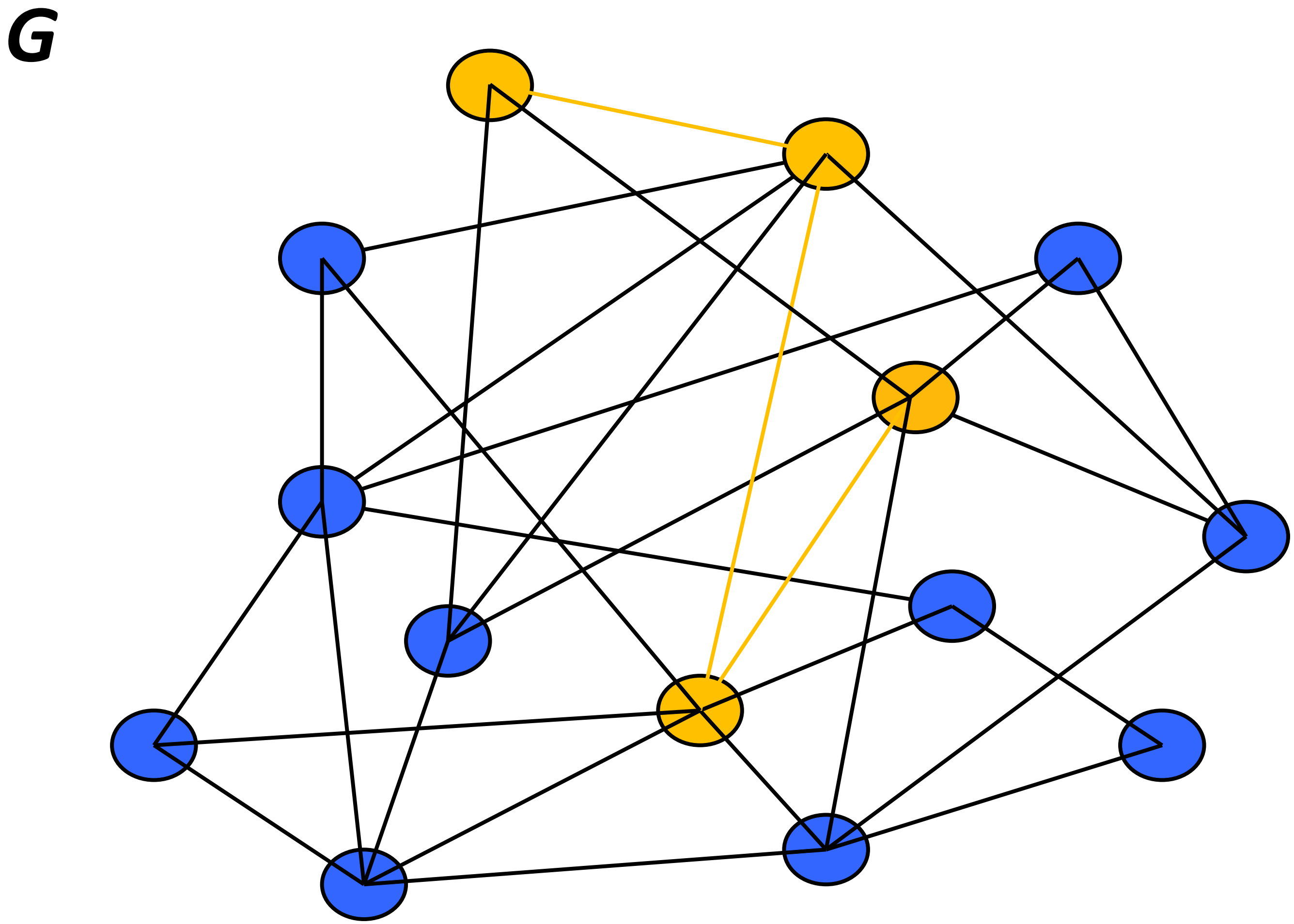


$v_2 \in N(v_1)$

The Random Walk Generator



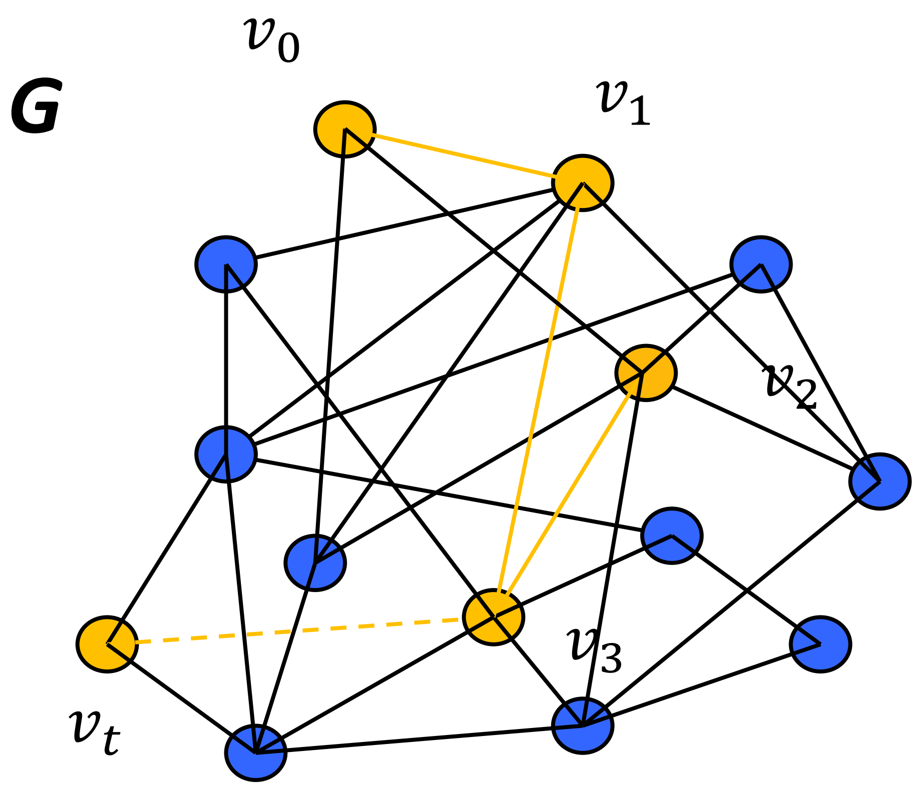
The Random Walk Generator



Formalizing the Problem

- Assume we will run the algorithm t+1 times. Start with truly random vertex u and take t random walk steps.
- Recall that X is the set of vertices on which the algorithm is not correct, we assume that $|X| \leq \frac{2^r}{100}$ (algorithm correct 99% of time)
- If at the end, we report the majority of the t+1 runs of algorithm, then we will return the correct answer as along as the random walk is inside X less than half the time.

The Random Walk Generator



T={o,…,t} time steps

S={i: $v_i \in X$}

We will show that

$$\Pr[|S| > t/2] \leq \left(\frac{2}{\sqrt{5}}\right)^{t+1}$$

Formalizing the Problem

- Initial distribution is uniform (start with truly random string): $\boldsymbol{p_0} = \mathbf{1}/n$
- Let χ_X and χ_Y the characteristic vectors of X and Y.
- Let $D_X = diag(X)$ and $D_Y = diag(Y)$
- Let $W = \frac{1}{d}A$ (not lazy) random walk matrix, with eigenvalues $\omega_1,\ldots, \omega_n$ such that $\omega_i \le \frac{1}{10}$ by the expansion requirement.
- Want to show $\Pr[|S| > t/2] \le (\frac{2}{\sqrt{5}})^{t+1}$

The Probability to be in X

- Fix a set $R \subseteq \{0, \dots, t\}$ of time steps.
- The probability that the walk is n X exacty during the steps in R is
  $$\Pr[Walk\ in\ X\ exactly\ for\ i \in R] = \langle 1, D_{Z_t} W \dots W D_{Z_0} p_0 \rangle$$
- Where $Z_i = X\ if\ i \in R\ \ and\ Y\ otherwise$
- Show that this probability is $\left(\frac{1}{5}\right)^{|R|}$.
- $\Pr[|S| > t/2] \leq (\frac{2}{\sqrt{5}})^{t+1}$ follows.

The Proof

- **Claim.**

  $$\Pr[Walk\ in\ X\ exactly\ for\ i \in R] = \langle 1, D_{Z_t}W \ldots WD_{Z_0}p_0 \rangle = \left(\frac{1}{5}\right)^{|R|}$$

- **Lemma.**

  $$||D_X W|| \leq 1/5.$$