CS 598: Spectral Graph Theory. Lecture 4

Graphic Inequalities and Lower bounds on λ_2

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- First Homework set is up (due February 12).
- You can work in groups of 2-3 but each one has to submit an individual write-up.
- Please give credit where credit is due.

Today

- Technique for proving inequalities on graphs, useful for graph approximations (future lectures)
- Use technique to lower-bound λ_2

Courant-Fischer for λ_2

$$\lambda_k = \min_{S \text{ of } \dim k} \max_{x \in S} \frac{x^T L x}{x^T x}$$

- Applying Courant-Fischer:

$$\lambda_2 = \min_{x \perp 1, x \neq 0} \frac{x^T L x}{x^T x} = \min_{x \perp 1, x \neq 0} \frac{\sum_{(i,j) \in E} (x_i - x_j)^2}{\sum_{i \in V} x_i^2}$$

- Useful for getting upper bounds
- To get upper bound on λ_2, just need to produce vector v with small Rayleigh Quotient.
- Every vector v provides an upper bound (test vector):

$$\lambda_2 \leq \frac{v^T L v}{v^T v}$$

Courant-Fischer for λ2

- Applying Courant-Fischer:

$$\lambda_k = \max_{S \text{ of } \dim n-k-1} \min_{x \in S} \frac{x^T L x}{x^T x}$$

$$\lambda_2 = \max_{S \text{ of } \dim n-1} \min_{v \in S} \frac{v^T L v}{v^T v}$$

- Not so useful as it is difficult to prove lower bounds on

$$\min \frac{v^T L v}{v^T v}$$

over a space of large dimension.

- Need new technique.

Graphic Inequalities

- When the symmetric matrix A is P.S.D we write $A \succcurlyeq 0$. Remember that A P.S.D means $v^T A v \geq 0$, for all v.
- Similarly, we write $A \succcurlyeq B$, or $A - B \succcurlyeq 0$, when $v^T A v \geq v^T B v$, for all v.
- $\preccurlyeq$ is partial order. It applies to some pairs of symmetric matrices and others are incomparable.

For the ones it does apply, we have:

$A \succcurlyeq B$ and B $\succcurlyeq C$ implies $A \succcurlyeq C$

$A \succcurlyeq B$ implies $A + C \succcurlyeq B + \mathrm{C}$

for symmetric A,B,C.

Graphic Inequalities

- Use same notation for graphs. I.e.

$$G \succcurlyeq H \quad \text{if} \quad L_G \succcurlyeq L_H$$

- Example: If G=(V,E) and H=(V,F) subgraph of G, then $L_G \succcurlyeq L_H$ (see blackboard)

Graphic Inequalities

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$$\mathrm{G} \succcurlyeq H \quad \text{if} \quad L_G \succcurlyeq L_H$$

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- Most useful when we consider some multiple of a graph: $\mathrm{G} \succcurlyeq c \cdot H, \ \mathrm{c} > 0$
- $c \cdot H$ is the same as H with the weight of each edge multiplied by c.

Graphic Inequalities

- **Lemma1:** Using Courant-Fischer we can show: If $\mathrm{G} \succcurlyeq c \cdot H$, then $\lambda_k(G) \succcurlyeq c\lambda_k(H)$

Proof: see blackboard

Graphic Inequalities

- **Lemma1:** Using Courant-Fischer we can show: If $\mathrm{G} \succcurlyeq c \cdot H$, then $\lambda_k(G) \succcurlyeq c\lambda_k(H)$

- **Lemma2:** Let G=(V,E,w) and H=(V,E,z) two graphs that differ only in their edge weights. Then

$$\mathrm{G} \succcurlyeq min_{e \in E} \frac{w(e)}{z(e)} \cdot H$$

Graph Approximations

- In the third part of this course, we will use the notion of one graph approximating another. This will mean that their Laplacian quadratic forms are similar. For example, H is a c-approximation of G if

$$cH \succcurlyeq \mathrm{G} \succcurlyeq H$$

- In general, we don't care that much about constants so we will say that a graph H is a c-approximation of G if there is a positive t for which

$$cH \succcurlyeq t\mathrm{G} \succcurlyeq H$$

- Later on, we will see that surprising approximations exist. For example, expanders are very sparse approximations of complete graph.

Graph Approximations: Examples

- How do we prove that $\mathrm{G} \succcurlyeq c \cdot H$ for some c and H?
- We first prove such inequalities for the simplest graphs, then extend to more general.
- **Lemma 1:** Let P_n be the path of length n-1 from vertex 1 to vertex n and $G_{1,n}$ the edge from 1 to n. Then:

$$(\mathrm{n}-1)P_n \succcurlyeq G_{1,n}$$

Graph Approximations: Examples

- We will show a more general lemma, for weighted paths

- Lemma 2: $G_{1,n} \preccurlyeq \left(\sum_{i=1}^{n-1} \frac{1}{w_i}\right) \sum_{i=1}^{n-1} w_i G_{i,i+1}$

Or, equivalently, $L_{(1,n)} \preccurlyeq \left(\sum_{i=1}^{n-1} \frac{1}{w_i}\right) \sum_{i=1}^{n-1} w_i L_{(i,i+1)}$

Proof: see blackboard

Bounding λ_2 of a Path Graph

- We obtain an upper bound and lower bound:

- Upper Bound from Lecture 2: $\lambda_2(P_n) \leq \frac{12}{n^2}$

  proof by exhibiting test vector v, such that v(i)=(n+1)-2i

- For lower bound, we use the "graph inequalities technique": prove that some multiple of the path is at least the complete graph (see blackboard):

$$\lambda_2(P_n) \geq \frac{4}{n^2}$$

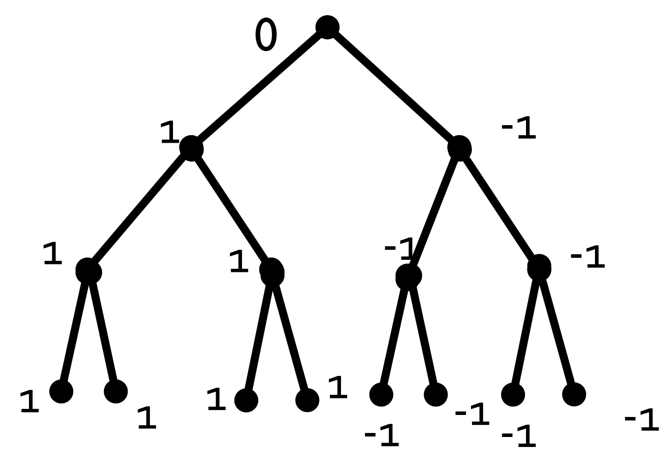
We will use: $(\mathrm{n}-1)P_n \succcurlyeq G_{1,n}$

Bounding λ2 of a Complete Binary Tree

- We obtain an upper bound and lower bound

- Upper bound from Lecture 2: $\lambda_2(T_n) \leq \dfrac{2}{n-1}$

  we used the test vector:



Bounding λ_2 of a Complete Binary Tree

- We obtain an upper bound and lower bound
- Upper bound from Lecture 2: $\lambda_2(T_n) \leq \dfrac{2}{n-1}$
- For lower bound, we use our technique, comparing the tree to the complete graph. (see blackboard)

$$\lambda_2(T_n) \geq \frac{1}{(n-1)\log_2 n}$$

Bounding λ_2 of a Complete Binary Tree

- We obtained an upper bound and lower bound:

$$\frac{1}{(n-1)\log_2 n} \leq \lambda_2(T_n) \leq \frac{2}{n-1}$$

- In Homework set you will show for some constant c:

$$\lambda_2(T_n) \geq \frac{1}{cn}$$

Bounding λ_2 of a Complete Binary Tree

- We obtained an upper bound and lower bound:

$$\frac{1}{(n-1)\log_2 n} \leq \lambda_2(T_n) \leq \frac{2}{n-1}$$

- Truth is

$$\frac{1}{n} < \lambda_2(T_n) < \frac{2}{n}$$

See: http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.45.5680